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The Equations of Radiation Hydrodynamics

In astrophysical flows, radiation often contains a large fraction of the energy density, momentum density, and stress (i.e., pressure) in the radiating fluid. Furthermore, radiative transfer is usually the most effective energy-exchange mechanism within the fluid. To describe the behavior of such flows we need conservation laws that account accurately for both the material and the radiative contributions to the flow dynamics.

To estimate the importance of radiation in fixing the *local* properties of a radiating fluid, consider the ratio R of the material internal energy density $\hat{e}$ to the radiation energy density E; for a perfect gas and equilibrium radiation

$$R \equiv \hat{e}/E = (3k/2a_R)(N/T^3) = 2.8 \times 10^{-2} N/T^3.$$

R also gives a measure of the relative importance of gas and radiation pressure because $p = \frac{2}{3}\hat{e}$ for a perfect gas, and $P = \frac{1}{3}E$ for radiation. Clearly radiation is most important at high temperatures and/or low densities. The two energy densities are about equal when $T_{\text{keV}} \approx 2\rho^{1/3}$, where T_{keV} is the temperature in kilovolts (1.2×10^7 K) and ρ is the material density in g cm^{-3}. Therefore, when temperatures reach a few keV (e.g., in X-ray sources or stellar interiors), radiation dominates the energy and pressure in the radiating fluid even at high densities.

In astrophysical systems R has a large range. For example $R \sim 10^4$ in the solar atmosphere, so radiative contributions to the energy density can be ignored; in an O-star atmosphere $R \sim 0.1$, and radiation is overwhelmingly important. This striking difference reflects both factor-of-ten larger temperatures and much lower densities in the O-star atmosphere compared to the Sun. Similarly, at the Sun's center $R \sim 500$, but at the center of an O-star $R \sim 1$. The large value of R in the solar interior reflects high densities (~ 100 g cm^{-3}) and temperatures of only about a kilovolt; in contrast the central temperature of an O-star is a few kilovolts and densities are a few g cm^{-3}.

The situation for energy *transport* in radiating flows is quite different. Radiative energy transfer usually dominates all other mechanisms even when temperatures are only about 1 eV and $E \ll \hat{e}$. In particular, radiative transport usually greatly exceeds thermal conduction because in equilibrium the photons and material particles have the same average energy, but photons travel at the speed of light, whereas material particles move only at about the sound speed; moreover, photons usually have much longer mean free paths than particles.

A semiquantitative measure of the relative importance of radiative and material energy transport in a radiating flow is given by the dimensionless *Boltzmann number*

$$\mathrm{Bo} \equiv (\rho c_p T v)/(\sigma_R T^4),$$

which is the ratio of the material enthalpy flux to the radiative flux from a free surface at temperature T. The Boltzmann number plays the same role for radiating fluids as the Peclet number does for nonradiating fluids [cf. (28.4)]. Recalling that $\sigma_R = \frac{1}{4} a_R c$, one sees that near a radiating surface Bo is of the order of (v/c) times the ratio defined above. In the solar atmosphere $(v/c) \sim 2\times10^{-5}$, and in an O-star atmosphere $(v/c) \sim 10^{-4}$, whence we conclude that radiative transport is dominant in the outer layers of most stars. In the interior of a star we must replace $\sigma_R T^4$ with $\sigma_R T_{\text{eff}}^4$, the *net* radiative flux; here energy transport by convection can dominate if the fluid moves at even a small fraction of its sound speed. If, on the other hand, the material is stable against convection (**C5,** Chap. 13), then radiative transport dominates in the interior as well.

Thus far we have discussed radiation as if it plays only an incidental role in a flow. But, in some cases, radiation can *drive* flows. For example, in the outer layers of a star radiative energy and momentum transport can drive or damp waves, drive stellar winds, and inhibit gravitational accretion. Furthermore, the temperature and density response of the opacity in the envelope of some stars allows radiation to drive stellar pulsations.

The equations of radiation hydrodynamics can be formulated in a variety of ways; each has advantages and disadvantages. One fundamental issue is whether to write the equations in an inertial frame fixed relative to an external observer (or the center of the star), or in the comoving fluid frame. Another concerns how best to describe the dynamical behavior of the radiation field. Thus in a stellar interior the radiation and material are in equilibrium, and we can treat the radiating fluid as a composite gas whose total energy, pressure, etc. are simple sums of the radiative and material contributions. But such an approach is virtually useless in the outer layers of a star where the radiation field has a strongly nonlocal character; here we must couple the dynamical equations to a full radiation transport equation.

In a moving fluid, the equation of transfer contains $O(v/c)$ frame-dependent terms that lead to similar terms in the dynamical equations for the radiating fluid. In contrast, the frame-dependent terms for a nonradiating fluid are only $O(v^2/c^2)$ (cf. §42). One can understand how $O(v/c)$ effects arise in a radiating fluid from simple classical considerations. First, there is an *advection* effect: a fluid element tends to "sweep up" ("leave behind") photons traveling against (along) its velocity vector, thus increasing (decreasing) the radiation energy density with which it can interact. Second, *Doppler shifts* affect the spectral distribution of the radiation field incident on the material. Consider a reference state with two fluid elements at rest, between which a certain energy and momentum exchange occurs. Now move one element relative to the other. Then, in addition to the change in the photon number density produced by advection, each photon will be blue (red) shifted, hence will have higher (lower) energy, when the two elements approach (recede from) one another. Both of these $O(v/c)$ effects can significantly affect the energy and momentum balance in a radiating fluid when the radiation field is intense.

The arguments advanced above are qualitative, and only serve to motivate a thorough mathematical analysis. In this work we will be guided by two precepts. First, we will pay close attention to the frame in which the equations are being written. In the past, failure to discriminate carefully between frames has led to confusion in the formulation of the dynamical equations, to misapplication of results valid in one frame to others in which they are not, and to serious conceptual errors. Second, we will retain mathematical consistency among various sets of equations to $O(v/c)$. The analysis is sometimes tedious, and may test our readers' patience. We assure them that this effort is not merely a quixotic obsession, but is essential to achieve equivalence among different forms of the radiating-fluid dynamical equations, both in a given frame, and between frames. The effort is vindicated by the surprising result that in certain regimes of interest, terms that are formally only $O(v/c)$ actually dominate over all others in the equations.

For didactic simplicity we ignore scattering and assume LTE. Though these restrictions afford considerable simplification, the resulting equations are complicated, and methods for solving them are not yet fully developed. Nevertheless it is essential to derive physically accurate equations, for it is clearly more useful to solve the correct equations, however approximately, than to solve incorrect equations, even exactly.

We first discuss (§7.1) the Lorentz transformation properties of quantities appearing in the transfer equation. In §7.2 we first write the transfer equation for moving media, then derive the energy and momentum equations for the radiating fluid (i.e., material plus radiation). We treat inertial-frame equations first because the derivation of the comoving-frame transfer equation is more complicated. We next discuss (§7.3) methods for solving these equations in one-dimensional flows. Here we consider first the important limiting case of diffusion, which offers penetrating insight into the dynamical behavior of the radiation field. We then discuss the comoving-frame equations, which are ideal for one-dimensional Lagrangean hydrodynamics calculations. Finally we consider two important versions of the inertial-frame equations.

7.1 Lorentz Transformation of the Transfer Equation

In order to write the transfer equation in different frames, we must determine the Lorentz transformation properties of its constituents: the specific intensity, opacity, emissivity, and photon directions and energies. In the formulae below the affix "0" denotes the comoving frame, in which material properties are isotropic.

89. The Photon Four-Momentum

In §37 we showed that the *photon four-momentum* is

$$M^\alpha = (h\nu/c)(1, \mathbf{n}) \tag{89.1}$$

where ν, $h\nu$, and $\mathbf{n}$ are the frequency, energy, and direction of propagation of the photon. The photon propagation four vector is

$$K^\alpha = (2\pi\nu/c)(1, \mathbf{n}). \tag{89.2}$$

Both K^α and M^α are null vectors.

The components of M^α in (89.1) are in Cartesian coordinates, hence are physical components. Later we will also need the contravariant components of M^α in spherical coordinates having a line element

$$ds^2 = -c^2\,dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{89.3}$$

Using equation (A3.41) we find

$$M^0 = h\nu/c, \tag{89.4a}$$

$$M^1 = (h\nu/c)\mu, \tag{89.4b}$$

$$M^2 = (h\nu/c)[(1-\mu^2)^{1/2}\cos\Phi]/r, \tag{89.4c}$$

and

$$M^3 = (h\nu/c)[(1-\mu^2)^{1/2}\sin\Phi]/(r\sin\theta), \tag{89.4d}$$

where $\Theta \equiv \cos^{-1}\mu$ and Φ are the polar and azimuthal angles of $\mathbf{n}$ relative to $\hat{\mathbf{r}}$.

If a photon has frequency ν and travels in direction $\mathbf{n}$ as measured in the lab frame, it will have some other frequency ν_0 and direction $\mathbf{n}_0$ as measured by an observer attached to a fluid element moving with velocity $\mathbf{v}$ relative to lab frame. Because M^α is a four-vector, its components in the two frames are related by the general Lorentz transformations (35.33) and (35.34), whence we obtain

$$\nu_0 = \gamma\nu(1 - \mathbf{n}\cdot\mathbf{v}/c) \tag{89.5}$$

and

$$\mathbf{n}_0 = (\nu/\nu_0)\{\mathbf{n} - \gamma(\mathbf{v}/c)[1 - (\gamma\mathbf{n}\cdot\mathbf{v}/c)/(\gamma+1)]\}, \tag{89.6}$$

or, equivalently,

$$\nu = \gamma\nu_0(1 + \mathbf{n}_0\cdot\mathbf{v}/c) \tag{89.7}$$

and

$$\mathbf{n} = (\nu_0/\nu)\{\mathbf{n}_0 + \gamma(\mathbf{v}/c)[1 + (\gamma\mathbf{n}_0\cdot\mathbf{v}/c)/(\gamma+1)]\}. \tag{89.8}$$

For the special case of motion along the z axis, (89.5) and (89.6) simplify to

$$(\nu_0, \nu_0 n_{x0}, \nu_0 n_{y0}, \nu_0 n_{z0}) = [\gamma\nu(1-n_z\beta), \nu n_x, \nu n_y, \gamma\nu(n_z-\beta)], \quad (89.9)$$

which implies

$$[\nu_0; \mu_0; (1-\mu_0^2)^{1/2}; \Phi_0] = [\gamma\nu(1-\beta\mu); (\mu-\beta)/(1-\beta\mu); (1-\mu^2)^{1/2}/\gamma(1-\beta\mu); \Phi]. \quad (89.10)$$

Similarly the inverse transformation gives

$$[\nu; \mu; (1-\mu^2)^{1/2}] = [\gamma\nu_0(1+\beta\mu_0); (\mu_0+\beta)/(1+\beta\mu_0); (1-\mu_0^2)^{1/2}/\gamma(1+\beta\mu_0)]. \quad (89.11)$$

Equations (89.10) and (89.11) describe the Doppler shift and aberration of light between frames in relative motion; the classical formulae are obtained by retaining terms only to $O(v/c)$, that is, by setting $\gamma \equiv 1$. These equations also apply to radial flow in spherical geometry.

From (89.10) one finds $d\nu_0 = (\nu_0/\nu)\,d\nu$, $d\mu_0 = (\nu/\nu_0)^2\,d\mu$, and $d\Phi = d\Phi_0$. Then recalling that $d\omega = d\mu\,d\Phi$ we see that $\nu\,d\nu\,d\omega$ is a Lorentz invariant:

$$\nu\,d\nu\,d\omega = \nu_0\,d\nu_0\,d\omega_0, \quad (89.12)$$

a result we will use repeatedly. Equation (89.12) has a deeper physical significance. In §43 we showed that for particles of any kind

$$d^3p/\tilde{e} = p^2\,dp\,d\omega/\tilde{e} \quad (89.13)$$

is an invariant. In particular, for photons $p = h\nu/c$ and $\tilde{e} = h\nu = cp$, hence the invariance of (89.13) implies (89.12).

90. Transformation Laws for the Specific Intensity, Opacity, and Emissivity

To determine the transformation properties of the specific intensity, we follow L. H. Thomas (**T1**) and calculate the number of photons N in a frequency interval $d\nu$, passing through an element of area dS oriented perpendicular to the z axis, into a solid angle $d\omega$ along an angle $\Theta = \cos^{-1}\mu$ to the z axis in a time interval dt. Let dS be stationary in the lab frame. Then

$$N = [I(\mu, \nu)/h\nu](d\omega\,d\nu)(dS\cos\Theta\,dt). \quad (90.1)$$

To an observer in a frame moving with velocity v along the z axis, dS appears to be moving with a velocity v in the negative z direction. This observer would therefore count

$$N_0 = [I_0(\mu_0, \nu_0)/h\nu_0](d\omega_0\,d\nu_0)[dS\cos\Theta_0\,dt_0 + (v/c)\,dS\,dt_0] \quad (90.2)$$

photons passing through dS; the first term gives the number of photons that would have been counted if dS had been stationary, while the second is the photon number density $\psi_0 = (I_0/ch\nu_0)$ times the volume $(dS\,v\,dt_0)$

swept out by dS in a time $dt_0 = \gamma\, dt$. But both observers must count the same *number* of photons passing through dS, hence $N = N_0$. Equating (90.1) and (90.2), and using (89.11) and (89.12) we find

$$I(\mu, \nu) = (\nu/\nu_0)^3 I_0(\mu_0, \nu_0). \tag{90.3}$$

That is, the quantity

$$\mathscr{I}(\mu, \nu) \equiv I(\mu, \nu)/\nu^3 \tag{90.4}$$

is a Lorentz invariant, called the *invariant intensity*.

We can obtain the same result by applying to the photon distribution function f_R the general arguments of §43, which led to the conclusion that the particle distribution function $f(\mathbf{x}, \mathbf{p}, t)$ is Lorentz invariant. From (63.4) we then immediately see that $\mathscr{I} = I/\nu^3 = \text{const.} \times f_R$ is an invariant.

Now consider the emissivity. Observers in all frames will count the same *number* of photons emitted from a definite volume element into a particular solid angle and frequency interval in a specified time interval. Hence

$$\eta(\mu, \nu)\, d\omega\, d\nu\, dV\, dt/h\nu = \eta_0(\nu_0)\, d\omega_0\, d\nu_0\, dV_0\, dt_0/h\nu_0. \tag{90.5}$$

Then using (89.12) and recalling that $dV\,dt$ is an invariant we find

$$\eta(\mu, \nu) = (\nu/\nu_0)^2 \eta_0(\nu_0), \tag{90.6}$$

where we noted that η is isotropic in the comoving frame.

Similarly, observers in all frames will count the same number of photons absorbed by a definite material element from a particular frequency interval and solid angle in a specified time interval. Hence

$$\chi(\mu, \nu) I(\mu, \nu)\, d\nu\, d\omega\, dV\, dt/h\nu = \chi_0(\nu_0) I_0(\mu_0, \nu_0)\, d\nu_0\, d\omega_0\, dV_0\, dt_0/h\nu_0, \tag{90.7}$$

whence

$$\chi(\mu, \nu) = (\nu_0/\nu)\chi_0(\nu_0). \tag{90.8}$$

We can also derive (90.8) from (90.3) and (90.6) by arguing that to achieve energy balance in equilibrium we must be able to equate the number of emissions and absorptions by a material element in all frames.

In deriving (90.3), (90.4), (90.6), and (90.8) we made use of the special Lorentz transformation for simplicity. The same results apply for arbitrary relative motion of the two frames provided that μ is replaced by $\mathbf{n}$, and we use (89.5) to (89.8) to relate $(\nu, \mathbf{n})$ to $(\nu_0, \mathbf{n}_0)$.

91. *The Radiation Stress-Energy Tensor and Four-Force Vector*

THE STRESS-ENERGY TENSOR

We now seek an expression for the *radiation stress-energy tensor* R, the spacetime generalization of the radiation stress tensor P defined in §65. We can infer the form of R by requiring that the space components R^{ij} be the rate of transport of the ith component of the radiative momentum per unit volume through a unit area oriented perpendicular to the jth coordinate axis. Thus we write

$$R^{ij} = \int f_R M^i c n^j \, d^3M \tag{91.1}$$

which is the integral of (number of particles per cm^3 per unit phase volume) $\times$ (momentum in i direction per particle) $\times$ (velocity component in j direction) over all phase space. But for a photon $cn^j = c^2M^j/\tilde{e}$, so we tentatively generalize (91.1) to

$$R^{\alpha\beta} = c^2 \int f_R M^\alpha M^\beta \frac{d^3M}{\tilde{e}}. \tag{91.2}$$

$R^{\alpha\beta}$ is obviously a four-tensor because it is the integral of the outer product of the four-vector M^α with itself, times the invariants f_R and $d^3M/\tilde{e}$.

We have already seen that the space components of (91.2) are the radiative stress. The component

$$R^{00} = \int f_R h\nu \, d^3M \tag{91.3}$$

is the integral of (number of particles per cm^3 per unit phase volume) $\times$ (energy per particle) over all phase space, and hence equals the radiation energy density. Likewise

$$R^{0i} = (1/c) \int f_R h\nu c n^i \, d^3M \tag{91.4}$$

equals $(1/c)$ times the energy flux density in the ith direction, while

$$R^{i0} = c \int f_R n^i (h\nu/c) \, d^3M \tag{91.5}$$

equals c times the momentum density in the ith direction. Thus R as given by (91.2) is a one-to-one analogue, for radiation, of the material stress-energy tensor defined in §40.

Note that (91.2) can also be applied to material particles, for which $p^i = mv^i$ and $\tilde{e} = mc^2$, where m is the relative mass of the particle. Thus (91.2) is the covariant generalization of the particle momentum flux density tensor (43.45), and provides a general expression for the stress-energy tensor in kinetic theory. The discussion above is purposely heuristic; a much deeper analysis that emphasizes the geometric aspects of the problem can be found in (**S6,** Chaps. 1–3).

Using (63.4) to replace f_R with the specific intensity, and noting that $p^2 \, dp \, d\omega = h^3\nu^2 \, d\nu \, d\omega/c^3$, we can write a continuum version of (91.2) as

$$R^{\alpha\beta} = c^{-1} \int_0^\infty d\nu \oint d\omega I(\mathbf{n}, \nu) n^\alpha n^\beta, \tag{91.6}$$

where we define $n^0 \equiv 1$ as in (89.2). $R^{\alpha\beta}$ as given by (91.6) is manifestly covariant because it is the outer product of the photon four-momentum with itself, times the invariants $I\nu^{-3}$ and $\nu\, d\nu\, d\omega$, integrated over all angles and frequencies. An equivalent form of (91.6) is

$$\mathsf{R} = \begin{pmatrix} E & c^{-1}\mathbf{F} \\ c^{-1}\mathbf{F} & \mathsf{P} \end{pmatrix}, \tag{91.7}$$

where E, $\mathbf{F}$, and P are the radiation energy density, flux, and stress tensor as defined in §§64 to 66. The elements of (91.7) can obviously be interpreted in exactly the same way as (91.3) to (91.5).

Using (66.6), one finds that in planar geometry (91.7) reduces to

$$\mathsf{R} = \begin{pmatrix} E & 0 & 0 & c^{-1}F \\ 0 & \frac{1}{2}(E-P) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(E-P) & 0 \\ c^{-1}F & 0 & 0 & P \end{pmatrix} \tag{91.8}$$

where $i = 1, 2, 3$ denote (x, y, z) respectively. The components in (91.8) are physical components, and are identical to the components measured with respect to an orthonormal tetrad in a curvilinear (e.g., spherical) coordinate system. Using the transformation rules (A3.47) we can write the contravariant components of R in spherical symmetry as

$$R^{\alpha\beta} = \begin{pmatrix} E & c^{-1}F & 0 & 0 \\ c^{-1}F & P & 0 & 0 \\ 0 & 0 & \frac{1}{2}(E-P)/r^2 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(E-P)/r^2 \sin^2\theta \end{pmatrix}, \tag{91.9}$$

where now $i = 1, 2, 3$ denote (r, θ, ϕ). Equations (91.7) to (91.9) also give the comoving-frame radiation stress-energy tensor R_0 if all quantities are measured in that frame.

The connection between R and R_0 is obtained from the Lorentz transformations (35.41) and (35.42). One finds

$$E = \gamma^2(E_0 + 2c^{-2}v_i F_0^i + c^{-2}v_i v_j P_0^{ij}), \tag{91.10}$$

$$F^i = \gamma\{F_0^i + \gamma E_0 v^i + v_j P_0^{ij} + [\gamma/c^2(\gamma+1)][(2\gamma+1)v_j F_0^j + \gamma v_j v_k P_0^{jk}]v^i\}, \tag{91.11}$$

and

$$\begin{aligned} P^{ij} = {} & P_0^{ij} + \gamma c^{-2}(v^i F_0^j + v^j F_0^i) + \gamma^2 c^{-2} E_0 v^i v^j \\ & + [\gamma^2/c^2(\gamma+1)](v^i v_k P_0^{jk} + v^j v_k P_0^{ki} + 2\gamma c^{-2} v_k F_0^k v^i v^j) \\ & + [\gamma^2/c^2(\gamma+1)]^2 (v_k v_l P_0^{kl}) v^i v^j. \end{aligned} \tag{91.12}$$

For one-dimensional flow in planar geometry, (91.10) to (91.12) reduce to

$$E=\gamma^2(E_0+2\beta c^{-1}F_0+\beta^2 P_0), \tag{91.13}$$

$$F=\gamma^2[(1+\beta^2)F_0+vE_0+vP_0], \tag{91.14}$$

and

$$P=\gamma^2(P_0+2\beta c^{-1}F_0+\beta^2 E_0). \tag{91.15}$$

These equations also apply in spherical symmetry for radial flow. We can further reduce (91.13) to (91.15) to $O(v/c)$, obtaining

$$E=E_0+2\beta c^{-1}F_0, \tag{91.16}$$

$$F=F_0+vE_0+vP_0, \tag{91.17}$$

and

$$P=P_0+2\beta c^{-1}F_0. \tag{91.18}$$

The corresponding inverse transformations are

$$(E_0, F_0, P_0)=[E-2\beta c^{-1}F, F-v(E+P), P-2\beta c^{-1}F]. \tag{91.19}$$

Equations (91.16) to (91.19) can also be derived by using (89.11), (89.12), and (90.3) expanded to first order in v/c. Thus $I_\nu\,d\nu\,d\omega=(\nu/\nu_0)^2 I^0_{\nu_0}\,d\nu_0\,d\omega_0\approx(1+2\beta\mu_0)I^0_{\nu_0}\,d\nu_0\,d\omega_0$, from which (91.16) follows by integrating over solid angle and frequency. Similarly, $\mu I_\nu\,d\nu\,d\omega=(\mu_0+\beta)(1+\beta\mu_0)I^0_{\nu_0}\,d\nu_0\,d\omega_0\approx[\mu_0+\beta(1+\mu_0^2)]I^0_{\nu_0}\,d\nu_0\,d\omega_0$ leads to (91.17), while $\mu^2 I_\nu\,d\nu\,d\omega=(\mu_0+\beta)^2 I^0_{\nu_0}\,d\nu_0\,d\omega_0\approx(\mu_0^2+2\beta\mu_0)I^0_{\nu_0}\,d\nu_0\,d\omega_0$ leads to (91.18). Note that (91.10) to (91.19) apply only to frequency-integrated moments.

THE FOUR-FORCE DENSITY VECTOR

By analogy with (42.1) we expect the dynamical equations for the radiation field to have the general form

$$R^{\alpha\beta}_{;\beta}=-G^\alpha, \tag{91.20}$$

where G^α is the *radiation four-force density* acting on the material. Thus the time component G^0 equals c^{-1} times the net rate of radiative energy input, per unit volume, into the matter, while the space components G^i equal the net rate of radiative momentum input. From these physical interpretations it is easy to write G^α in terms of macroscopic absorption and emission coefficients as

$$G^0=c^{-1}\int_0^\infty d\nu \oint d\omega[\chi(\mathbf{n},\nu)I(\mathbf{n},\nu)-\eta(\mathbf{n},\nu)] \tag{91.21a}$$

and

$$G^i=c^{-1}\int_0^\infty d\nu \oint d\omega[\chi(\mathbf{n},\nu)I(\mathbf{n},\nu)-\eta(\mathbf{n},\nu)]n^i. \tag{91.21b}$$

G^α is manifestly a four-vector, being the integral of the four-vector

$\nu(1, \mathbf{n})$, times the invariants $(\chi I/\nu^2)$ or (η/ν^2) and $\nu\, d\nu\, d\omega$, over all angles and frequencies. Thus (91.20), with R given by (91.7) and G^α by (91.21), is indeed a covariant conservation relation for the radiation field. For example, (91.20) in Cartesian coordinates yields the moment equations (78.4) and (78.11) derived in Chapter 6, consistent with the physical interpretation of those equations.

The relationship between G^α and G_0^α is obtained by Lorentz transformation. For one-dimensional flow in planar or spherical geometry,

$$G^0 = \gamma(G_0^0 + \beta G_0^1) \tag{91.22a}$$

and

$$G^1 = \gamma(G_0^1 + \beta G_0^0), \tag{91.22b}$$

or equivalently,

$$G_0^0 = \gamma(G^0 - \beta G^1) \tag{91.23a}$$

and

$$G_0^1 = \gamma(G^1 - \beta G^0). \tag{91.23b}$$

Here

$$cG^0 = 2\pi \int_0^\infty d\nu \int_{-1}^1 d\mu [\chi(\mu, \nu) I(\mu, \nu) - \eta(\mu, \nu)], \tag{91.24a}$$

$$cG^1 = 2\pi \int_0^\infty d\nu \int_{-1}^1 d\mu [\chi(\mu, \nu) I(\mu, \nu) - \eta(\mu, \nu)] \mu, \tag{91.24b}$$

$$cG_0^0 = \int_0^\infty [c\chi_0(\nu_0) E_0(\nu_0) - 4\pi\eta_0(\nu_0)]\, d\nu_0, \tag{91.25a}$$

and

$$cG_0^1 = \int_0^\infty \chi_0(\nu_0) F_0(\nu_0)\, d\nu_0. \tag{91.25b}$$

92. *Covariant Form of the Transfer Equation*

THE PHOTON BOLTZMANN EQUATION

For convenience, in this section we use units in which $h = c = 1$ and work in Cartesian coordinates. The standard Boltzmann equation for particles is

$$(\partial f/\partial t) + v^i(\partial f/\partial x^i) + p^i(\partial f/\partial p^i) = (Df/Dt)_{\text{coll}}. \tag{92.1}$$

An obvious covariant generalization of (92.1) is

$$\left(\frac{dx^\alpha}{d\tau}\right)\frac{\partial f}{\partial x^\alpha} + \left(\frac{dp^\alpha}{d\tau}\right)\frac{\partial f}{\partial p^\alpha} = \left(\frac{\delta f}{\delta \tau}\right)_{\text{coll}}, \tag{92.2}$$

where $(\delta/\delta\tau)$ is the intrinsic derivative with respect to proper time. Because photon world lines lie on the null cone, proper time is not a useful variable for the photon Boltzmann equation, so we replace τ by a new affine path-length variable ℓ defined such that

$$p^\alpha \equiv (dx^\alpha/d\ell). \tag{92.3}$$

[See (**S7,** §2.4) for a similar approach for geodesics, to which we return in §95.] We can then rewrite (92.2) as

$$p^\alpha(\partial f/\partial x^\alpha)+\dot{p}^\alpha(\partial f/\partial p^\alpha)=(\delta f/\delta \ell)_{\rm coll}, \tag{92.4}$$

where

$$\dot{p}^\alpha \equiv (dp^\alpha/d\ell). \tag{92.5}$$

For photons we identify p^α with M^α, and write the right-hand side in terms of a source ε and a sink $-\alpha f_R$, representing photon emission and absorption by the material. Thus the *photon Boltzmann equation* is

$$M^\alpha(\partial f_R/\partial x^\alpha)+\dot{M}^\alpha(\partial f_R/\partial M^\alpha)=\varepsilon-\alpha f_R, \tag{92.6}$$

or, in terms of the invariant intensity,

$$M^\alpha(\partial \mathscr{I}/\partial x^\alpha)+\dot{M}^\alpha(\partial \mathscr{I}/\partial M^\alpha)=e-\alpha\mathscr{I}. \tag{92.7}$$

Equation (92.7) applies in all frames, in particular in inertial frames. In the absence of general relativistic effects, photon trajectories in inertial frames are straight lines, hence $\dot{M}^\alpha\equiv 0$ (i.e., the photon four-momentum is conserved). Thus in an inertial frame (92.7) reduces to

$$M^\alpha \mathscr{I}_{,\alpha}=e-\alpha\mathscr{I}. \tag{92.8}$$

Substituting $\mathscr{I}=I/\nu^3$ and noting that ν is now a constant, we find that the left-hand side of (92.8) is ν^{-2} times the left-hand side of the time-dependent transfer equation (76.5). Therefore on the right-hand side we can identify

$$e=\eta_\nu/\nu^2 \tag{92.9a}$$

and

$$\alpha=\nu\chi_\nu. \tag{92.9b}$$

That is, e and α are just the invariant emissivity and invariant opacity discussed in §90.

LORENTZ INVARIANCE OF THE TRANSFER EQUATION

Let us now show that the transfer equation is covariant under Lorentz transformation between inertial frames. We stress that this statement holds only between frames moving *uniformly* relative to one another (see below and §95).

One approach is to argue that because $\mathscr{I}$ is a Lorentz invariant, $\mathscr{I}_{,\alpha}$ must be a covariant four-vector, hence $M^\alpha\mathscr{I}_{,\alpha}$ is an invariant. Thus between two inertial frames we can write

$$\begin{aligned}\frac{1}{\nu^2}[\eta(\mathbf{n},\nu)-\chi(\mathbf{n},\nu)I(\mathbf{n},\nu)]&=\frac{1}{\nu^2}\left[\frac{1}{c}\frac{\partial I(\mathbf{n},\nu)}{\partial t}+\mathbf{n}\cdot\nabla I(\mathbf{n},\nu)\right]\\ &=M^\alpha\mathscr{I}_{,\alpha}\equiv M'^\alpha\mathscr{I}'_{,\alpha}=\frac{1}{\nu'^2}\left[\frac{1}{c}\frac{\partial I'(\mathbf{n}',\nu')}{\partial t'}+\mathbf{n}'\cdot\nabla' I'(\mathrm{n}',\nu')\right].\end{aligned} \tag{92.10}$$

Equating the left- and right-most expressions in (92.10), and applying

(90.3), (90.6), and (90.8) we have

$$\frac{1}{c}\frac{\partial I'(\mathbf{n}', \nu')}{\partial t'} + \mathbf{n}' \cdot \nabla' I'(\mathbf{n}', \nu') = \eta'(\mathbf{n}', \nu') - \chi'(\mathbf{n}', \nu')I'(\mathbf{n}', \nu'), \quad (92.11)$$

which is identical in form to the transfer equation in the unprimed frame, as asserted.

Alternatively, we can use equations (35.39) and (35.12) to infer the transformation properties of the four-gradient (a covariant vector); for the special Lorentz transformation

$$\left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left[\gamma\left(\frac{1}{c}\frac{\partial}{\partial t'} - \beta\frac{\partial}{\partial z'}\right), \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \gamma\left(\frac{\partial}{\partial z'} - \frac{\beta}{c}\frac{\partial}{\partial t'}\right)\right]. \quad (92.12)$$

Combining (92.12) and (89.9) we then have

$$c^{-1}(\partial/\partial t) + (\mathbf{n} \cdot \nabla) \equiv (\nu'/\nu)[c^{-1}(\partial/\partial t') + (\mathbf{n}' \cdot \nabla')]. \quad (92.13)$$

Therefore

$$c^{-1}(\partial I_\nu/\partial t) + (\mathbf{n} \cdot \nabla) I_\nu = \eta_\nu - \chi_\nu I_\nu \quad (92.14)$$

transforms to

$$\begin{aligned}(\nu'/\nu)[c^{-1}(\partial/\partial t') + (\mathbf{n}' \cdot \nabla')][(\nu/\nu')^3 I'(\mathbf{n}', \nu')] \\ = (\nu/\nu')^2[\eta'(\mathbf{n}', \nu') - \chi'(\mathbf{n}', \nu')I'(\mathbf{n}', \nu')],\end{aligned} \quad (92.15)$$

and because ν/ν' is constant for *uniformly* moving frames, we recover (92.11).

NONINERTIAL FRAMES

When we transform from the lab frame to a *noninertial* frame such as the comoving frame of a fluid whose velocity varies in position and time, we can no longer take the ratio $(\nu/\nu')^3$ to be constant and remove it from the differential operator as we did in (92.15). Instead, new terms appear that account for changes in the Lorentz transformation from one point in the flow to another.

Put another way, a photon moving on a straight line with constant frequency in the lab frame suffers differing amounts of aberration and Doppler shift as measured in different fluid elements. Thus, in the ensemble of frames composing the comoving frame, we do *not* have $\dot{M}^\alpha \equiv 0$, and (92.8) ceases to be valid. Instead, we must start from (92.7) and generalize the transfer equation to an equation of the form

$$M^\alpha \mathscr{I}_{|\alpha} = e - a\mathscr{I} \quad (92.16)$$

where the operation $_{|\alpha}$ denotes a derivative taken subject to the constraint that photon paths remain on the null cone in the fluid frame. Equation (92.16) is the *Lagrangean transfer equation*, which we discuss in detail in §95.

7.2 The Dynamical Equations for a Radiating Fluid

We are now in a position to derive the dynamical equations for a radiating fluid. As our interest centers primarily on radiative effects, we will assume, for simplicity, that the material component is an ideal fluid; the effects of viscosity and heat conduction in the material can be included by using the results of Chapters 3 and 4.

We first develop an Eulerian formulation, in which all radiation quantities are measured in the laboratory frame, and both radiation and material properties are considered to be functions of $(\mathbf{x}, t)$. The Eulerian equations are conservation relations for the total (material plus radiation) energy and momentum in a *fixed* volume element. We can cast these equations into "quasi-Lagrangean" or "modified Eulerian" form by grouping time and space derivatives into the Lagrangean time derivative (D/Dt). However, the resulting equations are not truly Lagrangean because radiation quantities are measured in the lab, rather than comoving, frame; we develop the fully Lagrangean view in §§95 and 96.

The Eulerian equations are easier to apply in multidimensional flows; indeed, except in the diffusion approximation the Lagrangean equations have been used only for one-dimensional flows. On the other hand, complexities in the physics of the material properties and/or the radiation-material interaction are most easily handled in the Lagrangean frame; moveover the Lagrangean formulation often affords deeper physical insight.

93. The Inertial-Frame Transfer Equation for a Moving Fluid

Consider now the inertial-frame transfer equation for a moving medium, from which we will derive inertial-frame radiation energy and momentum equations. The main question that arises is how best to account for the Doppler shift and aberration of photons from the lab frame into the moving fluid frame, where they interact with the material.

In most astrophysical flows, v/c is so small that it is tempting to ask whether we could simply *ignore* velocity-dependent effects in calculating the radiation-material interaction (at least in the continuum where cross sections change slowly). This procedure has often been used; nevertheless we will shortly see that the answer is actually "no", and that we must retain the distinction between χ and χ_0, and η and η_0 to $O(v/c)$, and solve the transfer equation to this level of accuracy.

In principle we could solve the lab-frame transfer equation by brute force, using a large number of angles and frequencies and transforming these into the comoving frame via (89.5) to (89.11) when computing material absorption and emission coefficients. But this approach is unsatisfactory for two reasons. (1) The interaction terms are cumbersome double integrals over both angle and frequency [cf. (91.21)] that are costly to evaluate. (2) It obscures important underlying physics. We therefore seek other methods of treating the matter-radiation interaction.

FORMULATION

The simplest way to handle the lab-frame angle-frequency dependence of the absorption and emission terms is to use first-order expansions to evaluate the material coefficients at the appropriate fluid-frame frequency. That is, writing

$$\nu/\nu_0 = 1 + (\mathbf{n}\cdot\mathbf{v}/c), \tag{93.1}$$

equation (90.8) expanded to $O(v/c)$ yields

$$\chi(\mathbf{n},\nu) = \chi_0(\nu) - (\mathbf{n}\cdot\mathbf{v}/c)[\chi_0(\nu) + \nu(\partial\chi_0/\partial\nu)], \tag{93.2}$$

and (90.6) yields

$$\eta(\mathbf{n},\nu) = \eta_0(\nu) + (\mathbf{n}\cdot\mathbf{v}/c)[2\eta_0(\nu) - \nu(\partial\eta_0/\partial\nu)]. \tag{93.3}$$

Notice that in (93.2) and (93.3), ν is the *lab*-frame frequency of the radiation.

The *transfer equation* in Cartesian coordinates can then be written

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\mathbf{n},\nu)}{\partial t} + \mathbf{n}\cdot\nabla I(\mathbf{n},\nu) &= \eta_0(\nu) - \chi_0(\nu)I(\mathbf{n},\nu)\\ &+ \left(\frac{\mathbf{n}\cdot\mathbf{v}}{c}\right)\left\{2\eta_0(\nu) - \nu\frac{\partial\eta_0}{\partial\nu} + \left[\chi_0(\nu) + \nu\frac{\partial\chi_0}{\partial\nu}\right]I(\mathbf{n},\nu)\right\}.\end{aligned} \tag{93.4}$$

The advantage gained in this approach is that both χ_0 and η_0 are isotropic, which simplifies the calculation of angular moments of (93.4). While it is reasonable to expect (93.4) to be satisfactory for smooth continua, it will not be adequate for spectral lines because a first-order expansion in $\Delta\nu$ cannot accurately track the rapid variation of χ and η over a line profile, unless the velocity-induced frequency shifts are smaller than a line width (which is not the case for most problems of interest).

Integrating (93.4) over $d\omega$ we obtain the *monochromatic radiation energy equation*

$$(\partial E_\nu/\partial t) + (\partial F_\nu^i/\partial x^i) = 4\pi\eta_0(\nu) - c\chi_0(\nu)E_\nu + (v_i F_\nu^i/c)[\chi_0(\nu) + \nu(\partial\chi_0/\partial\nu)]. \tag{93.5}$$

Integrating (93.4) against $\mathbf{n}\,d\omega$ we obtain the *monochromatic radiation momentum equation*

$$\begin{aligned}c^{-2}(\partial F_\nu^i/\partial t) + (\partial P_\nu^{ij}/\partial x^j) &= -c^{-1}\chi_0(\nu)F_\nu^i + \tfrac{4}{3}\pi c^{-2}v^i[2\eta_0(\nu) - \nu(\partial\eta_0/\partial\nu)]\\ &+ c^{-1}v_j[\chi_0(\nu) + \nu(\partial\chi_0/\partial\nu)]P_\nu^{ij}.\end{aligned} \tag{93.6}$$

Here we noted that

$$\oint n^i n^j\,d\omega = \tfrac{4}{3}\pi\,\delta^{ij}. \tag{93.7}$$

Finally, integrating (93.5) and (93.6) over frequency we obtain the *radiation energy equation*

$$E_{,t}+F^i_{,i}=\int_0^\infty [4\pi\eta_0(\nu)-c\chi_0(\nu)E_\nu]\,d\nu + c^{-1}v_i\int_0^\infty [\chi_0(\nu)+\nu(\partial\chi_0/\partial\nu)]F^i_\nu\,d\nu=-cG^0 \tag{93.8}$$

and the *radiation momentum equation*

$$c^{-2}F^i_{,t}+P^{ij}_{,j}=-c^{-1}\int_0^\infty \chi_0(\nu)F^i_\nu\,d\nu+4\pi c^{-2}v^i\int_0^\infty \eta_0(\nu)\,d\nu + c^{-1}v_j\int_0^\infty [\chi_0(\nu)+\nu(\partial\chi_0/\partial\nu)]P^{ij}_\nu\,d\nu=-G^i. \tag{93.9}$$

It is important to notice that the first terms on the right-hand sides of (93.8) and (93.9) are *not* cG_0^0 and G_0^i as defined in (91.25), despite their superficial resemblance. In G_0^0 and G_0^i *all* quantities are evaluated in the comoving frame; in contrast, in (93.8) and (93.9) the material coefficients are in the comoving frame while radiation quantities and frequencies are in the inertial frame. To call attention to this combination of frames we refer to (93.4) to (93.9) as *mixed-frame equations.*

To obtain the corresponding equations in spherical symmetry we merely replace the left-hand sides of (93.4) to (93.6), (93.8), and (93.9) with the left-hand sides of (76.9), (78.5), (78.6), (78.13), and (78.14) because only the interaction terms are affected by the expansion procedure. Scattering terms are complicated in the mixed-frame equations; we therefore ignore them and set $\chi\equiv\kappa$ for the remainder of §93. A detailed discussion of scattering is given in (**F2**) [see also (**M8**)].

ON THE IMPORTANCE OF $O(v/c)$ TERMS

Let us now examine the physical importance of the v/c terms in (93.8) and (93.9). To simplify the discussion we specialize to grey material:

$$E_{,t}+F^i_{,i}=\kappa(4\pi B-cE)+(\kappa/c)v_iF^i=-cG^0 \tag{93.10}$$

and

$$c^{-2}F_{,t}+P^{ij}_{,j}=(\kappa/c)[-F^i+v^i(4\pi B/c)+v_jP^{ij}]=-G^i. \tag{93.11}$$

Consider first the energy equation. We instantly see that if we omit the $O(v/c)$ terms, we lose a term equal to the rate of work done by the radiation force on the material, a serious error when the radiation field is intense. Furthermore, in the diffusion regime $E_0\rightarrow(4\pi B/c)$, hence from (91.16) and (91.17) $4\pi B-cE=-2\mathbf{v}\cdot\mathbf{F}/c+O(v^2/c^2)$. Equation (93.10) then becomes

$$E_{,t}+F^i_{,i}=-(\kappa/c)v_iF^i, \tag{93.12}$$

which is essentially the first law of thermodynamics for the radiation field. It states that the rate of change of the radiation energy density in a fixed volume plus the rate of work done by the radiation force on the material equals the net rate of (radiant) heat influx through the boundary surface of the volume. Thus in the diffusion regime we reach three important conclusions. (1) Omission of the $O(v/c)$ terms from the radiation energy equation produces an error equal in size to ignoring the net absorption-emission term, which is unacceptable. (2) We arrive at the physically correct statement (93.12) only by retaining $O(v/c)$ terms. (3) Dimensional analysis suggests that $\kappa \mathbf{v} \cdot \mathbf{F}/c$ is $O(lv/\lambda_p c)$ relative to $\nabla \cdot \mathbf{F}$; hence the velocity-dependent term may actually dominate the energy balance in the dynamic diffusion regime where $v/c \gtrsim \lambda_p/l$.

Next consider the momentum equation in the diffusion regime. On a fluid-flow time scale the time derivative is only $O(\lambda_p v/lc)$ relative to $\kappa F/c$, hence is negligible. We can therefore write

$$F^i = -(c/\kappa)P^{ij}_{,j} + v^i(4\pi B/c) + v_j P^{ij}, \tag{93.13}$$

In §97 we will show that in the diffusion limit $E_0 \rightarrow (4\pi B/c)$, $P^{ij} \rightarrow P_0^{ij} + O(\lambda_p v/lc) = \frac{1}{3}E_0\,\delta^{ij} + O(\lambda_p v/lc)$, and $\mathbf{F}_0 \rightarrow -(c/\kappa)\nabla \cdot \mathsf{P}_0$. Thus to $O(v/c)$ equation (93.13) reduces to

$$F^i = F_0^i + v^i E_0 + v_j P_0^{ij} = F_0^i + \tfrac{4}{3} v^i E_0, \tag{93.14}$$

which is just the Lorentz transformation from $\mathbf{F}_0$ to $\mathbf{F}$, cf. (91.17). Hence if we were to omit $O(v/c)$ terms in (93.11) we would fail to discriminate between the inertial-frame (Eulerian) and the comoving-frame (Lagrangean) radiation flux. To appreciate the importance of this point, recall from §80 that in a stellar interior $(vE_0/F_0) \sim (v/c)(T/T_{\text{eff}})^4 \sim 10^{12}(v/c)$, which implies that even a minuscule velocity produces a huge difference between $\mathbf{F}$ and $\mathbf{F}_0$. In short, the $O(v/c)$ terms in (93.11) are *essential* if we are to obtain the correct lab-frame flux in a moving fluid.

RELATIVE SIZES OF TERMS

The thrust of the discussion above is that terms that are formally $O(v/c)$, and which therefore appear, at first sight, to be negligible can sometimes dominate all others in the equation. Hence we must undertake a detailed analysis of the relative sizes of terms in (93.4) to (93.9) in all regimes of interest. In the streaming limit we consider both radiation-flow and fluid-flow time scales; in the diffusion limit we consider both static and dynamic diffusion.

We assume that $(v/c) \ll 1$, and agree that terms that are *always* of $O(v/c)$ or smaller relative to the dominant terms can be dropped. The key word here is "always" because terms that are negligible in one regime may dominate in another, and because any real flow spans both the optically thin and thick limits. As we desire our calculations to be accurate in both limits and successfully bridge the gap between, *any term found to be essential in one regime must be retained in all regimes.*

In the streaming limit $\lambda_p/l \gtrsim 1$, $E \approx P$, and $F \approx cE$. In the diffusion limit $E = 3P$. For static diffusion $t_f \gg t_d$ and $(v/c) \ll (\lambda_p/l)$; in this case the first term on the right-hand side of (91.17) dominates and $F \rightarrow F_0$, hence F/cE is $O(\lambda_p/l)$. For dynamic diffusion $t_f \lesssim t_d$ and $(v/c) \gtrsim (\lambda_p/l)$; in this case the last two terms in (91.17) dominate, and F/cE is $O(v/c)$. Similarly the net absorption-emission term [i.e., $\kappa(cE - 4\pi B)$] is $O(c\lambda_p/l^2)E$ for static diffusion (cf. §80), and $O(v/l)E$ for dynamic diffusion (cf. §97).

Consider first the transfer equation (93.4). In the streaming regime, dimensional analysis suggests that on a fluid-flow time scale the five terms in the equation scale as $(v/c):1:(l/\lambda_p):(l/\lambda_p):(v/c)(l/\lambda_p)$. Here we can drop both the time derivative (the radiation field is quasi static) and the velocity-dependent term on the right-hand side, retaining only the spatial operator and the absorption-emission terms. For radiation flow on a time scale t_R, the $(\partial/\partial t)$ term becomes $O(1)$ and must be retained. Now consider the diffusion regime, grouping the net emission $\eta - \kappa I$ into a single term. For static diffusion the terms scale as $(v/c):1:(\lambda_p/l):(v/c)(l/\lambda_p)$; for dynamic diffusion they scale as $(v/c):1:(v/c):(v/c)(l/\lambda_p)$. In both cases the time derivative can be dropped. For dynamic diffusion the velocity-dependent term may actually dominate all others in the equation. Even for static diffusion it will dominate the net absorption-emission term if $(v/c) \gtrsim (\lambda_p/l)^2$. Inasmuch as we always retain the absorption-emission terms, we must retain the velocity-dependent term as well. In short, *to obtain a correct solution of the inertial-frame transfer equation on a fluid-flow time scale we must retain the spatial operator on the left-hand side of* (93.4), *and all terms on the right-hand side.* To follow radiation flow on a time scale t_R, we must also retain the time derivative.

Next consider the radiation energy equation (93.10), starting with the streaming limit. Dimensional analysis suggests that on a fluid-flow time scale the five terms in the equation scale as $(v/c):1:(l/\lambda_p):(l/\lambda_p):(v/c)(l/\lambda_p)$; thus we need retain only $\nabla \cdot \mathbf{F}$ and the absorption-emission terms. To follow radiation flow, we also need to retain $(\partial/\partial t)$, which becomes $O(1)$ on a time scale t_R. An exceptional case arises if the medium is nearly in radiative equilibrium; here the absorption-emission terms may cancel almost exactly, and $(\partial/\partial t)$ and the velocity-dependent terms can then fix the energy blance. In this event we must retain all terms in the equation. Now consider the static diffusion limit. Here the terms scale as $(v/c)(l/\lambda_p):1:1:(v/c)(l/\lambda_p)$, where the net absorption-emission terms are grouped together. In this regime we can drop both the $(\partial/\partial t)$ and velocity-dependent terms because $(v/c) \ll (\lambda_p/l)$. But when $(v/c) \rightarrow (\lambda_p/l)$, all terms in the equation become of the same order and must be retained. In the dynamic diffusion limit the terms scale as $1:1:1:(v/c)(l/\lambda_p)$; here the velocity-dependent term may dominate all others.

Finally, consider the radiation momentum equation (93.9), starting with the streaming limit. On a fluid-flow time scale the terms scale as $(v/c):1:(l/\lambda_p):(v/c)(l/\lambda_p):(v/c)(l/\lambda_p)$. We need retain only $\nabla \cdot \mathsf{P}$ and the integral over $\mathbf{F}$, all other terms being at most $O(v/c)$. On a radiation-flow time scale we must also retain $(\partial/\partial t)$. In the static diffusion limit, the terms scale as $(v/c)(\lambda_p/l):1:1:(v/c)(l/\lambda_p):(v/c)(l/\lambda_p)$. In this regime we can drop both the time-derivative and velocity-dependent terms. But as $(v/c) \rightarrow (\lambda_p/l)$, the velocity-dependent terms become of the same order as $\nabla \cdot \mathsf{P}$ and must be retained, while $(\partial/\partial t)$ is only $O(v^2/c^2)$. Finally, in the dynamic diffusion limit, the terms scale as $(v^2/c^2):1:(v/c)(l/\lambda_p):(v/c)(l/\lambda_p):(v/c)(l/\lambda_p)$. Here we can drop $(\partial/\partial t)$, but must retain $\nabla \cdot \mathsf{P}$, and all three terms on the right-hand side, which are of the same size and may actually dominate the solution [cf. discussion of (93.14)].

In summary, *to solve the inertial-frame radiation energy and momentum equations correctly on a fluid-flow time scale we must retain all terms in both equations except $(\partial/\partial t)$ in the momentum equation, which can be dropped.* To follow radiation flow we must retain $(\partial/\partial t)$ in the momentum equation as well. Unfortunately, these requirements make the equations cumbersome to solve.

94. *Inertial-Frame Equations of Radiation Hydrodynamics*

The radiation energy and momentum equations discussed in §93 are to be solved simultaneously with conservation equations for the material, which we now derive.

GENERAL FORM

The dynamical equations for the radiation field can be written (cf. §91)

$$R^{\alpha\beta}_{;\beta} = -G^{\alpha}. \tag{94.1}$$

This expression is manifestly covariant and applies in all frames. In an *inertial* frame the covariant derivative can be evaluated immediately in any coordinate system, using the formulae in §A3. In a *noninertial* frame, we must first construct the spacetime metric before we can compute the Christoffel symbols needed to evaluate the covariant derivative of the stress-energy tensor (see §95).

In Cartesian coordinates, substitution of (91.7) into (94.1) immediately yields the radiation energy equation

$$E_{,t} + F^{i}_{,i} = -cG^{0} \tag{94.2}$$

and the radiation momentum equation

$$c^{-2}F^{i}_{,t} + P^{ij}_{,j} = -G^{i}, \tag{94.3}$$

where G^0 and G^i are given in general by (91.21), or to $O(v/c)$ by (93.8) and (93.9). In spherical symmetry we can apply equation (A3.89) to (91.9) or (A3.91) to (91.8), noting that only $(\partial/\partial t)$ and $(\partial/\partial r)$ are nonvanishing, to obtain

$$(\partial E/\partial t)+r^{-2}[\partial(r^2F)/\partial r]=-cG^0 \tag{94.4}$$

and

$$c^{-2}(\partial F/\partial t)+(\partial P/\partial r)+(3P-E)/r=-G^1. \tag{94.5}$$

To obtain dynamical equations for a radiating fluid we use a similar approach, adopting either of two equivalent physical pictures. On one hand, we can consider the radiation field as providing an additional four-force acting on the material, and modify the dynamical equations for the material to read

$$M^{\alpha\beta}_{;\beta}=F^\alpha+G^\alpha. \tag{94.6}$$

Alternatively we can consider the externally imposed four-force F^α to act on a radiating fluid, comprising matter plus radiation, which has a total stress-energy tensor

$$S^{\alpha\beta}=M^{\alpha\beta}+R^{\alpha\beta}; \tag{94.7}$$

we then obtain the dynamical equations

$$(M^{\alpha\beta}+R^{\alpha\beta})_{;\beta}=F^\alpha. \tag{94.8}$$

In view of (94.1), equations (94.6) and (94.8) are mathematically equivalent. As we will see, (94.8) provides a conceptually more satisfying formulation in the diffusion regime, whereas (94.6) is more natural in the streaming limit.

Writing (94.6) and (94.8) in Cartesian coordinates for an ideal material fluid plus radiation, we obtain the relativistically correct equations

$$(\rho_1c^2-p)_{,t}+(\rho_1c^2v^j)_{,j}=v_jf^j+cG^0 \tag{94.9a}$$

and

$$(\rho_1v_i)_{,t}+(\rho_1v_iv^j)_{,j}=f_i-p_{,i}+G_i, \tag{94.10a}$$

or

$$(\rho_1c^2-p+E)_{,t}+(\rho_1c^2v^j+F^j)_{,j}=v_jf^j \tag{94.9b}$$

and

$$(\rho_1v^i+c^{-2}F^i)_{,t}+(\rho_1v^iv^j+P^{ij})_{,j}=f^i-\delta^{ij}p_{,j}. \tag{94.10b}$$

Here $\rho_1\equiv\gamma^2\rho_{000}$, ρ_{000} is defined by (40.9), and f^i is the Newtonian force density. Comparable expressions for general three-dimensional flows in spherical coordinates are given in (**P3,** 230–231).

If we subtract c^2 times the continuity equation (39.8) from (94.9) we obtain

$$[(\gamma-1)\rho c^2+\gamma\rho e+(\gamma^2-1)p]_{,t}+\{[(\gamma-1)\rho c^2+\gamma\rho e+\gamma^2p]v^i\}_{,i}=v_if^i+cG^0 \tag{94.11a}$$

or

$$[(\gamma-1)\rho c^2+\gamma\rho e+(\gamma^2-1)p+E]_{,t}+\{[(\gamma-1)\rho c^2+\gamma\rho e+\gamma^2 p]v^i+F^i\}_{,i}=v_i f^i, \tag{94.11b}$$

which will prove useful below; here, as in (39.5), $\rho\equiv\gamma\rho_0$.

The flows with which we deal are nonrelativistic; let us therefore reduce (94.9) to (94.11) to expressions correct to $O(v/c)$.

THE MOMENTUM EQUATION

We can cast the momentum equations (94.10) into a simpler form (as we did for a nonradiating fluid in §42) by multiplying (94.9) by v^i/c^2 and subtracting from (94.10) to obtain the relativistically correct equations (**W2**):

$$\rho_*(D\mathbf{v}/D\tau)=\mathbf{f}-\nabla p-c^2\mathbf{v}(p_{,t}+\mathbf{v}\cdot\mathbf{f})+\mathbf{G}-c^{-1}\mathbf{v}G^0, \tag{94.12a}$$

or

$$\rho_*\frac{D\mathbf{v}}{D\tau}=\mathbf{f}-\nabla p-\frac{\mathbf{v}}{c^2}\left(\frac{\partial p}{\partial t}+\mathbf{v}\cdot\mathbf{f}\right)-\left(\nabla\cdot\mathsf{P}+\frac{1}{c^2}\frac{\partial\mathbf{F}}{\partial t}\right)+\frac{\mathbf{v}}{c^2}\left(\frac{\partial E}{\partial t}+\nabla\cdot\mathbf{F}\right). \tag{94.12b}$$

Here $\rho_*\equiv\gamma\rho_{000}$, and $\mathbf{G}$ denotes the space components of G^α.

In §42 we saw that the distinctions between t and τ, and ρ and ρ_*, are $O(v^2/c^2)$, as is $\mathbf{v}(p_{,t}+\mathbf{v}\cdot\mathbf{f})/c^2$ relative to other terms. Hence for a nonradiating fluid, the Newtonian momentum equation is correct to $O(v/c)$. In contrast, for a radiating fluid the frame-dependent term $\mathbf{v}G^0/c$ can be $O(v/c)$ relative to the radiation force $\mathbf{G}$ in the streaming limit, hence the radiating-fluid momentum equation correct to $O(v/c)$ is

$$\rho(D\mathbf{v}/Dt)=\mathbf{f}-\nabla p+\mathbf{G}-(\mathbf{v}/c)G^0 \tag{94.13a}$$

or

$$\rho(D\mathbf{v}/Dt)=\mathbf{f}-\nabla p-[c^{-2}(\partial\mathbf{F}/\partial t)+\nabla\cdot\mathsf{P}]+c^{-2}\mathbf{v}[(\partial E/\partial t)+\nabla\cdot\mathbf{F}]. \tag{94.13b}$$

On a fluid-flow time scale the term containing $(\partial E/\partial t)$ is $O(v^2/c^2)$ relative to $\nabla\cdot\mathsf{P}$ and can be dropped. These equations are quasi-Lagrangean in the sense defined earlier.

The first two terms on the right-hand side of (94.13) account for externally imposed and pressure gradient forces. The third term accounts for the radiation force, expressed either as the momentum absorbed by the material from the radiative flux, or as the divergence of the radiation pressure tensor. The last term accounts for changes in the equivalent mass density of the material, as measured in the lab frame, resulting from any net gain or loss of energy by the material through its interaction with the radiation field. This term has often been omitted in discussions of radiation hydrodynamics [see, e.g., equation (9.83) in (**P3**)], but at a sacrifice in logical consistency. In particular, we will see in §96 that it is essential to retain this term in order to make an exact correspondence between the inertial-frame and comoving-frame momentum equations for a radiating fluid.

While granting the *logical* importance of the $O(v/c)$ terms in (94.13), we have agreed that because $(v/c) \ll 1$ we can drop terms that are always of this order, or smaller, for practical computations. In (94.13a), $\mathbf{v}G^0/c$ is at most $O(v/c)$ relative to $\mathbf{G}$ in the streaming limit, and even smaller if the material is in radiative equilibrium. In the diffusion limit $\mathbf{v}G^0/c$ is $O(\lambda_p v/lc)$ or $O(v^2/c^2)$ relative to $\mathbf{G}$ in the static and dynamic diffusion regimes respectively. Thus in all cases this term may be dropped. Similarly, in (94.13b), F/c is $O(1)$ relative to P in the streaming limit, and is $O(\lambda_p/l)$ or $O(v/c)$ relative to P in the static or dynamic diffusion limits; hence on a fluid-flow time scale both terms containing $\mathbf{F}$ are at most $O(v/c)$ relative to $\nabla \cdot \mathsf{P}$ and can be dropped. Thus the inertial-frame momentum equations suited to practical computation are

$$\rho(D\mathbf{v}/Dt) = \mathbf{f} - \nabla p + \mathbf{G} \tag{94.14a}$$

or

$$\rho(D\mathbf{v}/Dt) = \mathbf{f} - \nabla p - \nabla \cdot \mathsf{P}, \tag{94.14b}$$

that is, the standard Newtonian equations of motion including a radiative force.

Expressions for (94.13b) in spherical geometry, with the v/c term omitted, are given in (**P3,** 231).

THE TOTAL ENERGY EQUATION

To obtain the total energy equation for a nonrelativistic radiating fluid we simply let $\gamma \to 1$ and $(\gamma - 1) \to \frac{1}{2}v^2/c^2$ in (94.11). We then have

$$(\rho e + \tfrac{1}{2}\rho v^2)_{,t} + \{[\rho(e + \tfrac{1}{2}v^2) + p]v^i\}_{,i} = v_i f^i + cG^0, \tag{94.15a}$$

or

$$(\rho e + \tfrac{1}{2}\rho v^2 + E)_{,t} + \{[\rho(e + \tfrac{1}{2}v^2) + p]v^i + F^i\}_{,i} = v_i f^i. \tag{94.15b}$$

These Eulerian equations are correct to $O(v/c)$. Equation (94.15a) states that the rate of change of the material energy (internal plus kinetic) in a fixed volume equals the rate of work done by external forces and fluid stresses, plus the net rate of energy input to the material by absorption and emission of radiation, minus the net flux of material energy through the surface bounding the volume. Similarly, integrating (94.15b) over a fixed volume element and applying the divergence theorem, we obtain the statement that the rate of change of the total energy (internal, kinetic, and radiative) in the volume equals the rate of work done on the element by external forces and fluid stresses, minus the flux of total energy (material plus radiative) out of the volume. Detailed expressions for (94.15b) in spherical coordinates are given in (**P3,** 232).

Using (19.13) we can rewrite (94.15) in the quasi-Lagrangean form

$$\rho D(e+\tfrac{1}{2}v^2)/Dt+\nabla\cdot(p\mathbf{v})=\mathbf{v}\cdot\mathbf{f}+cG^0 \qquad (94.16a)$$

or

$$\rho D(e+\tfrac{1}{2}v^2)/Dt+(\partial E/\partial t)+\nabla\cdot(p\mathbf{v}+\mathbf{F})=\mathbf{v}\cdot\mathbf{f}. \qquad (94.16b)$$

These equations will prove useful later.

THE MECHANICAL ENERGY EQUATION

To obtain a mechanical energy equation for a radiating fluid, we form the dot product of (94.13) with $\mathbf{v}$ and drop terms of $O(v^2/c^2)$, which yields

$$\rho D(\tfrac{1}{2}v^2)/Dt=-\mathbf{v}\cdot(\nabla p)+\mathbf{v}\cdot(\mathbf{f}+\mathbf{G}) \qquad (94.17a)$$

or

$$\rho D(\tfrac{1}{2}v^2)/Dt=-\mathbf{v}\cdot(\nabla p)+\mathbf{v}\cdot\mathbf{f}-\mathbf{v}\cdot[c^{-2}(\partial\mathbf{F}/\partial t)+\nabla\cdot\mathsf{P}\,]. \qquad (94.17b)$$

These (quasi-Lagrangean) equations state that the rate of change of the kinetic energy per unit mass in a material element equals the rate of work, per unit mass, done by applied external and radiative forces, minus the work done against fluid stresses.

On a fluid-flow time scale the $(\partial\mathbf{F}/\partial t)$ term in (94.17b) is at most $O(v/c)$ relative to $\nabla\cdot\mathsf{P}$; hence this term is $O(v^2/c^2)$ overall and can be dropped. On a radiation-flow time scale this term is of the same order as $\nabla\cdot\mathsf{P}$ in the streaming limit.

THE GAS-ENERGY EQUATION

In §42 we derived the relativistically correct gas-energy equation for a nonradiating fluid. By exactly the same analysis, using (94.6), (94.9a), and (94.10a) we find that the relativistic gas-energy equation for a radiating fluid is

$$\rho_0\left[\frac{De}{D\tau}+p\frac{D}{D\tau}\left(\frac{1}{\rho_0}\right)\right]=-V_\alpha F^\alpha-V_\alpha G^\alpha. \qquad (94.18)$$

As before, $V_\alpha F^\alpha\equiv 0$, while $-V_\alpha G^\alpha=\gamma(cG^0-\mathbf{v}\cdot\mathbf{G})$. The inner product $V_\alpha G^\alpha$ is not zero for radiation as it is for ordinary body forces because the radiant energy absorbed by the material produces a change in its total proper energy (cf. §37). Recalling that $(dt/d\tau)=\gamma$, we see that the lab-frame gas-energy equation for a radiating fluid is

$$\rho_0\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho_0}\right)\right]=cG^0-\mathbf{v}\cdot\mathbf{G} \qquad (94.19a)$$

or

$$\rho_0\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho_0}\right)\right]=-\left(\frac{\partial E}{\partial t}+\nabla\cdot\mathbf{F}\right)+\mathbf{v}\cdot\left(\frac{1}{c^2}\frac{\partial\mathbf{F}}{\partial t}+\nabla\cdot\mathsf{P}\right). \qquad (94.19b)$$

Recalling (91.23a), we see that (94.19a) is the first law of thermodynamics for matter in the presence of radiation. It states that the rate of change of the internal energy per unit mass in a material element plus the rate of mechanical work done by the material in expansion, equals the net rate, per unit mass, of "heat" input from the radiation field, evaluated in the comoving fluid frame (cf. §96); compare with (93.12).

We emphasize that in (94.19) all radiation quantities are measured in the lab frame, while the material properties e, p, and ρ_0 are all measured in the comoving frame. But for the latter the distinction between frames is $O(v^2/c^2)$ and hence can be ignored to $O(v/c)$. Thus (94.19) could also be derived simply by taking the difference between the $O(v/c)$ equations (94.16) and (94.17).

Dimensional analysis suggests that in (94.19a) $\mathbf{v} \cdot \mathbf{G}$ is of the same order as cG^0 in the dynamic diffusion regime, and may exceed cG^0 in the streaming limit if the material is approximately in radiative equilibrium. Hence both terms on the right-hand side must be retained. In (94.19b), $c^{-2}\mathbf{v} \cdot (\partial\mathbf{F}/\partial t)$ is $O(v^2/c^2)$ relative to $\nabla \cdot \mathbf{F}$ on a fluid-flow time scale, and hence can be dropped. The remaining three terms are all of the same order in the dynamic diffusion regime, hence all must be retained. Thus for practical calculations the inertial-frame gas-energy equation is

$$\rho\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right]=cG^0-\mathbf{v}\cdot\mathbf{G} \tag{94.20a}$$

or

$$\rho\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right]=-\nabla\cdot\mathbf{F}-\frac{\partial E}{\partial t}+\mathbf{v}\cdot\nabla\cdot\mathsf{P}. \tag{94.20b}$$

Equation (94.20b) can be rewritten in either the Eulerian form

$$(\rho e+E)_{,t}+\nabla\cdot[(\rho e+p)\mathbf{v}+\mathbf{F}]=\mathbf{v}\cdot(\nabla p+\nabla\cdot\mathsf{P}) \tag{94.21}$$

or, using (19.13), in the quasi-Lagrangean form

$$\rho\left[\frac{D}{Dt}\left(e+\frac{E}{\rho}\right)+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right]+\nabla\cdot(\mathbf{F}-\mathbf{v}E)=\mathbf{v}\cdot\nabla\cdot\mathsf{P}. \tag{94.22}$$

By straightforward manipulation (94.22) can be recast as

$$\rho\left[\frac{D}{Dt}\left(e+\frac{E}{\rho}\right)+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)+\frac{1}{\rho}\,\mathsf{P}:\nabla\mathbf{v}\right]+\nabla\cdot(\mathbf{F}-\mathbf{v}E-\mathbf{v}\cdot\mathsf{P})=0; \tag{94.23}$$

compare with (96.9). Here $\mathsf{P}:\nabla\mathbf{v}$ denotes the contraction $P^{ij}v_{i,j}$.

COUPLING TO THE RADIATION EQUATIONS

The radiating-fluid momentum and energy equations written above are to be solved simultaneously with the radiation energy and momentum equations of §93 [i.e. (93.8) and (93.9) or perhaps (93.10) and (93.11)]. When simplified for practical computations on fluid-flow time scales, the fluid momentum, total energy, and mechanical energy equations are all standard Newtonian equations which include radiative terms in exactly the way one would expect from heuristic arguments. Only the gas energy equation contains a velocity-dependent radiation term that would be unanticipated from simple Newtonian arguments; this term, often ignored in inertial-frame formulations of the equations of radiation hydrodynamics, is required to convert the net rate of radiant energy input into the material to its value in the comoving fluid frame (cf. §96). Thus the *fluid* equations contain few surprises (the exception being the gas energy equation) and can be handled in the usual way. In contrast, it is in the *radiation* energy and momentum equations that special care is required, for, as we have seen, it is essential that *all* velocity-dependent terms be retained if we are to obtain the correct radiation energy and momentum balance. It is at this juncture that most Eulerian-frame treatments of radiation hydrodynamics are flawed, for the velocity-dependent terms are usually dropped, and the radiation equations are treated as if the material is at rest, which is simply incorrect.

95. *The Comoving-Frame Equation of Transfer*

RATIONALE FOR THE COMOVING FRAME

In radiation hydrodynamics the *comoving frame* of a fluid parcel comprises a *set* of inertial frames, each of which has a velocity that instantaneously coincides with that of the parcel. Clearly this frame is identical to the *Lagrangean frame* of fluid dynamics, and further is the *proper frame* in the relativistic sense, and is therefore the frame in which microscopic descriptions of material properties by thermodynamics and statistical mechanics apply. It is also the frame in which details of the interaction between radiation and matter (e.g., partial redistribution by scattering) are most easily handled (**M13**). Moreover, it offers computational advantages because it is the frame in which material properties are isotropic, and in which the frequency mesh can be tailored to describe accurately the absorption spectrum of the material; the latter point is especially important in line-formation problems (**M11**). Thus the comoving frame is the natural frame for one-dimensional flow problems such as stellar pulsations, and is the frame always used (whether explicitly or implicitly) in stellar evolution calculations that invoke the diffusion-limit solution of the transfer equation.

Because the velocity field in a flow is, in general, a function of both position and time, the comoving frame associated with any particular fluid element is a noninertial frame. Photon trajectories in the comoving frame are therefore not Euclidian straight lines, but are *geodesics* whose shapes are determined by the metric of the curved (i.e., non-Minkowskian) spacetime through which the photons move. In addition, photon frequencies are not constant in this spacetime. As a result, the comoving-frame transfer equation is more complicated than the lab-frame equation, and contains derivatives with respect to angle and frequency in addition to space coordinates and time.

There are two routes by which the comoving-frame equation of transfer can be derived, each having certain advantages. In the first we use special relativity in an inertial spacetime to derive an equation correct to all orders in (v/c); the results can then be reduced to $O(v/c)$. At this point one can safely invoke Galilean relativity because all $O(v/c)$ terms have been accounted for, and all remaining special relativistic terms are $O(v^2/c^2)$ or higher; hence a fullly Lagrangean formulation can be constructed simply by grouping terms to form the Lagrangean time derivative (D/Dt). Alternatively we can derive the equation in a noninertial Lagrangean frame from the outset, using the techniques of general relativity; here we obtain results accurate only to $O(v/c)$, but enjoy a more direct hold on the physics and deeper insight into the geometrical aspects of the problem. We will develop both approaches, limiting the discussion to one-dimensional spherically symmetric flows.

The main goal of §§95 and 96 is to obtain equations in which all *physical* variables, for both radiation and matter, are expressed in the Lagrangean frame. But we emphasize that this choice of frame is critical only for the *dependent* variables, and that the choice of *grid* (i.e., *independent* variables) on which the equations are to be solved is a matter of complete indifference. Indeed we may choose Eulerian coordinates fixed in space, Lagrangean coordinates fixed in the fluid (§98), or a *freely moving* coordinate system that is neither [e.g., an *adaptive mesh* that moves both in inertial space and with respect to fluid elements (**T3**), (**W3**)]. In practice the adaptive-mesh schemes have proven to be extraordinarily powerful tools in solving astrophysical radiation-hydrodynamics problems.

SPECIAL RELATIVISTIC FORMULATION

In deriving relativistic equations of hydrodynamics, we expressed the material stress-energy tensor in terms of proper quantities and calculated derivatives in an inertial spacetime. We can do the same for radiation, obtaining a transfer equation containing intensities, material properties, angles, and frequencies in the comoving frame only.

The inertial-frame transfer equation for spherically symmetric flow is

$$\frac{1}{c}\frac{\partial I(\mu,\nu)}{\partial t}+\mu\frac{\partial I(\mu,\nu)}{\partial r}+\frac{(1-\mu^2)}{r}\frac{\partial I(\mu,\nu)}{\partial \mu}=\eta(\mu,\nu)-\chi(\mu,\nu)I(\mu,\nu). \tag{95.1}$$

Using (90.3), (90.6), and (90.8) we can rewrite (95.1) as

$$\begin{aligned}\left(\frac{\nu}{\nu_0}\right)&\left[\frac{1}{c}\frac{\partial I_0(\mu_0,\nu_0)}{\partial t}+\mu\frac{\partial I_0(\mu_0,\nu_0)}{\partial r}+\frac{(1-\mu^2)}{r}\frac{\partial I_0(\mu_0,\nu_0)}{\partial \mu}\right]\\ &-3\left(\frac{\nu}{\nu_0^2}\right)\left[\frac{1}{c}\frac{\partial \nu_0}{\partial t}+\mu\frac{\partial \nu_0}{\partial r}+\frac{(1-\mu^2)}{r}\frac{\partial \nu_0}{\partial \mu}\right]I_0(\mu_0,\nu_0)\\ &=\eta_0(\nu_0)-\chi_0(\nu_0)I_0(\mu_0,\nu_0).\end{aligned} \tag{95.2}$$

When the derivatives in (95.2) are calculated, it is assumed that both μ and ν are held constant (with the exception of $\partial/\partial\mu$, of course). Because the fluid velocity varies in space and time, the comoving-frame quantities μ_0 and ν_0 are not constant, and we must account for their variations.

To calculate the derivatives of $I_0(\mu_0, \nu_0)$ we apply the chain rules

$$\left.\frac{\partial}{\partial t}\right|_{r\mu\nu} = \left.\frac{\partial}{\partial t}\right|_{r\mu_0\nu_0} + \left.\frac{\partial \mu_0}{\partial t}\right|_{r\mu\nu}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial t}\right|_{r\mu\nu}\frac{\partial}{\partial \nu_0}, \tag{95.3}$$

$$\left.\frac{\partial}{\partial r}\right|_{t\mu\nu} = \left.\frac{\partial}{\partial r}\right|_{t\mu_0\nu_0} + \left.\frac{\partial \mu_0}{\partial r}\right|_{t\mu\nu}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial r}\right|_{t\mu\nu}\frac{\partial}{\partial \nu_0}, \tag{95.4}$$

and

$$\left.\frac{\partial}{\partial \mu}\right|_{rt\nu} = \left.\frac{\partial \mu_0}{\partial \mu}\right|_{rt\nu}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial \mu}\right|_{rt\nu}\frac{\partial}{\partial \nu_0}. \tag{95.5}$$

By repeated use of equations (89.10) and (89.11) one can evaluate all the derivatives written above in terms of comoving-frame quantities only; one finds

$$(\partial\mu_0/\partial t) = -\gamma^2(1-\mu_0^2)(\partial\beta/\partial t), \tag{95.6a}$$

$$(\partial\nu_0/\partial t) = -\gamma^2\mu_0\nu_0(\partial\beta/\partial t), \tag{95.6b}$$

$$(\partial\mu_0/\partial r) = -\gamma^2(1-\mu_0^2)(\partial\beta/\partial r), \tag{95.7a}$$

$$(\partial\nu_0/\partial r) = -\gamma^2\mu_0\nu_0(\partial\beta/\partial r), \tag{95.7b}$$

$$(\partial\mu_0/\partial\mu) = \gamma^2(1+\beta\mu_0)^2, \tag{95.8a}$$

and

$$(\partial\nu_0/\partial\mu) = -\beta\gamma^2(1+\beta\mu_0)\nu_0. \tag{95.8b}$$

Substituting (95.3) to (95.8) into (95.2) we find, after some reduction, the *comoving-frame transfer equation*

$$\begin{aligned}
&\frac{\gamma}{c}(1+\beta\mu_0)\frac{\partial I_0(\mu_0,\nu_0)}{\partial t} + \gamma(\mu_0+\beta)\frac{\partial I_0(\mu_0,\nu_0)}{\partial r} \\
&\quad + \frac{\partial}{\partial\mu_0}\left\{\gamma(1-\mu_0^2)\left[\frac{(1+\beta\mu_0)}{r} - \gamma^2(\mu_0+\beta)\frac{\partial\beta}{\partial r} - \frac{\gamma^2}{c}(1+\beta\mu_0)\frac{\partial\beta}{\partial t}\right]I_0(\mu_0,\nu_0)\right\} \\
&\quad - \frac{\partial}{\partial\nu_0}\left\{\gamma\nu_0\left[\frac{\beta(1-\mu_0^2)}{r} + \gamma^2\mu_0(\mu_0+\beta)\frac{\partial\beta}{\partial r} + \frac{\gamma^2}{c}\mu_0(1+\beta\mu_0)\frac{\partial\beta}{\partial t}\right]I_0(\mu_0,\nu_0)\right\} \\
&\quad + \gamma\left\{\frac{2\mu_0+\beta(3-\mu_0^2)}{r} + \gamma^2(1+\mu_0^2+2\beta\mu_0)\frac{\partial\beta}{\partial r}\right. \\
&\quad \left. + \frac{\gamma^2}{c}[2\mu_0+\beta(1+\mu_0^2)]\frac{\partial\beta}{\partial t}\right\}I_0(\mu_0,\nu_0) = \eta_0(\nu_0) - \chi_0(\nu_0)I_0(\mu_0,\nu_0).
\end{aligned} \tag{95.9}$$

We have grouped terms so that the angle and frequency derivatives are in *conservative form* (i.e., such that they vanish when integrated over their full ranges). Equation (95.9) is valid for $0 \leq |\beta| < 1$, and hence can be used in relativistic flows.

Integrating (95.9) over comoving-frame angles we obtain frequency-dependent moment equations. Define

$$Q_0(\nu_0) \equiv 2\pi \int_{-1}^{1} I_0(\mu_0, \nu_0)\mu_0^3 \, d\mu_0. \tag{95.10}$$

Then integrating (95.9) against $d\omega_0/4\pi$ we obtain the *monochromatic radiation energy equation*

$$\begin{aligned}
&\gamma\left[\frac{\partial E_0(\nu_0)}{\partial t} + \frac{v}{c^2}\frac{\partial F_0(\nu_0)}{\partial t}\right] + \gamma\left[\frac{\partial F_0(\nu_0)}{\partial r} + v\frac{\partial E_0(\nu_0)}{\partial r}\right] \\
&\quad + \gamma\left\{\frac{1}{r}[2F_0(\nu_0) + 3vE_0(\nu_0) - vP_0(\nu_0)] + \gamma^2 \frac{\partial v}{\partial r}\left[E_0(\nu_0) + P_0(\nu_0) + \frac{2v}{c^2}F_0(\nu_0)\right]\right. \\
&\quad \left. + \frac{\gamma^2}{c^2}\frac{\partial v}{\partial t}[2F_0(\nu_0) + vE_0(\nu_0) + vP_0(\nu_0)]\right\} \\
&\quad - \frac{\partial}{\partial \nu_0}\left[\!\!\left[\gamma\nu_0\left\{\frac{v}{r}[E_0(\nu_0) - P_0(\nu_0)] + \gamma^2\frac{\partial v}{\partial r}\left[P_0(\nu_0) + \frac{v}{c^2}F_0(\nu_0)\right]\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{\gamma^2}{c^2}\frac{\partial v}{\partial t}[F_0(\nu_0) + vP_0(\nu_0)]\right\}\right]\!\!\right] = 4\pi\eta_0(\nu_0) - c\chi_0(\nu_0)E_0(\nu_0).
\end{aligned} \tag{95.11}$$

Integrating (95.9) against $\mu_0 \, d\omega_0/4\pi$, we obtain the *monochromatic radiation momentum equation*

$$\begin{aligned}
&\frac{\gamma}{c^2}\left[\frac{\partial F_0(\nu_0)}{\partial t} + v\frac{\partial P_0(\nu_0)}{\partial t}\right] + \gamma\left[\frac{\partial P_0(\nu_0)}{\partial r} + \frac{v}{c^2}\frac{\partial F_0(\nu_0)}{\partial r}\right] \\
&\quad + \gamma\left\{\frac{1}{r}\left[3P_0(\nu_0) - E_0(\nu_0) + \frac{2v}{c^2}F_0(\nu_0)\right] + \frac{\gamma^2}{c^2}\frac{\partial v}{\partial r}[2F_0(\nu_0) + vE_0(\nu_0) + vP_0(\nu_0)]\right. \\
&\quad \left. + \frac{\gamma^2}{c^2}\frac{\partial v}{\partial t}\left[E_0(\nu_0) + P_0(\nu_0) + \frac{2v}{c^2}F_0(\nu_0)\right]\right\} \\
&\quad - \frac{\partial}{\partial \nu_0}\left[\!\!\left[\gamma\nu_0\left\{\frac{v}{c^2 r}[F_0(\nu_0) - Q_0(\nu_0)] + \frac{\gamma^2}{c^2}\frac{\partial v}{\partial r}[Q_0(\nu_0) + vP_0(\nu_0)]\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{\gamma^2}{c^2}\frac{\partial v}{\partial t}\left[P_0(\nu_0) + \frac{v}{c^2}Q_0(\nu_0)\right]\right\}\right]\!\!\right] = -\frac{\chi_0(\nu_0)}{c}F_0(\nu_0).
\end{aligned} \tag{95.12}$$

Note that these equations contain *four* moments, unlike the inertial-frame equations in which Q does not appear.

Integrating (95.11) and (95.12) over comoving-frame frequency we obtain the *radiation energy equation*

$$\gamma\left(\frac{\partial E_0}{\partial t}+\frac{v}{c^2}\frac{\partial F_0}{\partial t}\right)+\gamma\left(\frac{\partial F_0}{\partial r}+v\frac{\partial E_0}{\partial r}\right)+\gamma\left[\frac{1}{r}(2F_0+3vE_0-vP_0)\right.$$
$$\left.+\gamma^2\frac{\partial v}{\partial r}\left(E_0+P_0+\frac{2v}{c^2}F_0\right)+\frac{\gamma^2}{c^2}\frac{\partial v}{\partial t}(2F_0+vE_0+vP_0)\right] \qquad (95.13)$$
$$=\int_0^\infty[4\pi\eta_0(\nu_0)-c\chi_0(\nu_0)E_0(\nu_0)]\,d\nu_0,$$

and the *radiation momentum equation*

$$\frac{\gamma}{c^2}\left(\frac{\partial F_0}{\partial t}+v\frac{\partial P_0}{\partial t}\right)+\gamma\left(\frac{\partial P_0}{\partial r}+\frac{v}{c^2}\frac{\partial F_0}{\partial r}\right)+\gamma\left[\frac{1}{r}\left(3P_0-E_0+\frac{2v}{c^2}F_0\right)\right.$$
$$\left.+\frac{\gamma^2}{c^2}\frac{\partial v}{\partial r}(2F_0+vE_0+vP_0)+\frac{\gamma^2}{c^2}\frac{\partial v}{\partial t}\left(E_0+P_0+\frac{2v}{c^2}F_0\right)\right] \qquad (95.14)$$
$$=-\frac{1}{c}\int_0^\infty\chi_0(\nu_0)F_0(\nu_0)\,d\nu_0.$$

Notice that the third moment Q_0 has vanished from these equations.

To check (95.13) and (95.14), start with the inertial-frame radiation energy and momentum equations

$$(\partial E/\partial t)+(\partial F/\partial r)+2F/r=-cG^0 \qquad (95.15)$$

and

$$c^{-2}(\partial F/\partial t)+(\partial P/\partial r)+(3P-E)/r=-G^1, \qquad (95.16)$$

and use (91.13) to (91.15) to eliminate (E, F, P) in favor of (E_0, F_0, P_0), and (91.22) to express G^0 and G^1 in terms of G_0^0 and G_0^1. One then finds that (95.15) equals (95.13) plus β times (95.14), and that (95.16) equals (95.14) plus β times (95.13) [cf. (**M4**)]. We are thus assured of exact consistency between the inertial- and comoving-frame equations.

Equations (95.9) to (95.14) apply in the high-velocity limit and hence can be used to describe radiative transfer in, say, the cosmic expansion, supernova blast waves, and other high-velocity flows. But for most flows $(v/c)\ll 1$ and it suffices to work only to $O(v/c)$. To first order in β, the transfer equation reduces to

$$\frac{1}{c}\frac{DI_0(\mu_0,\nu_0)}{Dt}+\frac{\mu_0}{r^2}\frac{\partial}{\partial r}[r^2I_0(\mu_0,\nu_0)]$$
$$+\frac{\partial}{\partial\mu_0}\left\{(1-\mu_0^2)\left[\frac{1}{r}+\frac{\mu_0}{c}\left(\frac{v}{r}-\frac{\partial v}{\partial r}\right)-\frac{a}{c^2}\right]I_0(\mu_0,\nu_0)\right\}$$
$$-\frac{\partial}{\partial\nu_0}\left\{\nu_0\left[(1-\mu_0^2)\frac{v}{cr}+\frac{\mu_0^2}{c}\frac{\partial v}{\partial r}+\frac{\mu_0 a}{c^2}\right]I_0(\mu_0,\nu_0)\right\} \qquad (95.17)$$
$$+\left[(3-\mu_0^2)\frac{v}{cr}+\frac{(1+\mu_0^2)}{c}\frac{\partial v}{\partial r}+\frac{2\mu_0 a}{c^2}\right]I_0(\mu_0,\nu_0)$$
$$=\eta_0(\nu_0)-\chi_0(\nu_0)I_0(\mu_0,\nu_0).$$

Here $a = (\partial v/\partial t)$, the fluid acceleration. In (95.17) we have grouped terms to form the Lagrangean time derivative (D/Dt) and have written the spatial derivative in conservative form.

Similarly, the monochromatic radiation energy equation to $O(v/c)$ is

$$\begin{aligned}\frac{DE_0(\nu_0)}{Dt}&+\frac{1}{r^2}\frac{\partial}{\partial r}[r^2F_0(\nu_0)]+\frac{v}{r}[3E_0(\nu_0)-P_0(\nu_0)]\\&+\frac{\partial v}{\partial r}[E_0(\nu_0)+P_0(\nu_0)]+\frac{2a}{c^2}F_0(\nu_0)\\&-\frac{\partial}{\partial \nu_0}\left[\!\left[\nu_0\left\{\frac{v}{r}[E_0(\nu_0)-P_0(\nu_0)]+\frac{\partial v}{\partial r}P_0(\nu_0)+\frac{a}{c^2}F_0(\nu_0)\right\}\right]\!\right]\\&=4\pi\eta_0(\nu_0)-c\chi_0(\nu_0)E_0(\nu_0),\end{aligned}\tag{95.18}$$

and the monochromatic radiation momentum equation is

$$\begin{aligned}\frac{1}{c^2}\frac{DF_0(\nu_0)}{Dt}&+\frac{\partial P_0(\nu_0)}{\partial r}+\frac{3P_0(\nu_0)-E_0(\nu_0)}{r}\\&+\frac{2}{c^2}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)F_0(\nu_0)+\frac{a}{c^2}[E_0(\nu_0)+P_0(\nu_0)]\\&-\frac{\partial}{\partial \nu_0}\left[\!\left[\nu_0\left\{\frac{v}{c^2r}[F_0(\nu_0)-Q_0(\nu_0)]+\frac{1}{c^2}\frac{\partial v}{\partial r}Q_0(\nu_0)+\frac{a}{c^2}P_0(\nu_0)\right\}\right]\!\right]\\&=-\frac{1}{c}\chi_0(\nu_0)F_0(\nu_0).\end{aligned}\tag{95.19}$$

Finally, the radiation energy equation to $O(v/c)$ is

$$\begin{aligned}\frac{DE_0}{Dt}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_0)+\frac{v}{r}(3E_0-P_0)+\frac{\partial v}{\partial r}(E_0+P_0)+\frac{2a}{c^2}F_0&\\=\int_0^\infty[4\pi\eta_0(\nu_0)-c\chi_0(\nu_0)E_0(\nu_0)]\,d\nu_0&,\end{aligned}\tag{95.20}$$

and the radiation momentum equation is

$$\begin{aligned}\frac{1}{c^2}\frac{DF_0}{Dt}+\frac{\partial P_0}{\partial r}+\frac{3P_0-E_0}{r}+\frac{2}{c^2}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)F_0+\frac{a}{c^2}(E_0+P_0)&\\=-\frac{1}{c}\int_0^\infty\chi_0(\nu_0)F_0(\nu_0)\,d\nu_0&.\end{aligned}\tag{95.21}$$

Equations (95.17) to (95.21) are equivalent to those derived by Castor (**C3**) and Buchler (**B2**), except that Castor omits the acceleration terms. On a fluid-flow time scale these terms are $O(v/c)$ compared to those in (v/r) or $(\partial v/\partial r)$, hence $O(v^2/c^2)$ overall and can be dropped; however if the velocity evolves on a radiation-flow time scale they should be retained. Moreover, as we will see in §97, these terms have an interesting physical significance.

The planar limits of (95.17) to (95.21) and (95.9) to (95.14) are obtained by letting $(1/r)\to 0$. Buchler (**B2**), (**B3**) also gives results for cylindrical geometry.

Finally, to demonstrate explicitly the consistency between the inertial- and comoving-frame dynamical equations for radiation, consider a grey, planar, pure-absorbing medium in LTE. Equations (95.20) and (95.21) become (omitting acceleration terms)

$$(\partial E_0/\partial t)+(\partial F_0/\partial z)+v(\partial E_0/\partial z)+(\partial v/\partial z)(E_0+P_0)=\kappa_0(4\pi B_0-cE_0), \tag{95.22}$$

and

$$c^{-2}(\partial F_0/\partial t)+(\partial P_0/\partial z)+(v/c^2)(\partial F_0/\partial z)+(2/c^2)(\partial v/\partial z)F_0=-c^{-1}\kappa_0 F_0. \tag{95.23}$$

On the other hand, using (91.16) to (91.18), we can rewrite the inertial-frame equation (93.10)

$$(\partial E/\partial t)+(\partial F/\partial z)=\kappa_0(4\pi B_0-cE)+(v/c)\kappa_0 F \tag{95.24}$$

as

$$\frac{\partial E_0}{\partial t}+\frac{\partial F_0}{\partial z}+v\left(\frac{\partial E_0}{\partial z}+\frac{\partial P_0}{\partial z}\right)+\frac{\partial v}{\partial z}(E_0+P_0)=\kappa_0(4\pi B_0-cE_0)-\frac{v}{c}\kappa_0 F_0+O\left(\frac{v^2}{c^2}\right). \tag{95.25}$$

Regrouping terms and using (95.23) we find

$$\begin{aligned}\frac{DE_0}{Dt}+\frac{\partial F_0}{\partial z}+\frac{\partial v}{\partial z}(E_0+P_0)&=\kappa_0(4\pi B_0-cE_0)-v\left(\frac{\kappa_0}{c}F_0+\frac{\partial P_0}{\partial z}\right)\\&=\kappa_0(4\pi B_0-cE_0)+O\left(\frac{v^2}{c^2}\right),\end{aligned} \tag{95.26}$$

which is identical to (95.22). Similarly the inertial-frame equation (93.11)

$$c^{-2}(\partial F/\partial t)+(\partial P/\partial z)=(\kappa_0/c)[-F+(4\pi v/c)B_0+vP] \tag{95.27}$$

becomes

$$\frac{1}{c^2}\frac{\partial F_0}{\partial t}+\frac{\partial P_0}{\partial z}+\frac{2v}{c^2}\frac{\partial F_0}{\partial z}+\frac{2}{c^2}\frac{\partial v}{\partial z}F_0=-\frac{\kappa_0 F_0}{c}+\frac{v}{c^2}\kappa_0(4\pi B_0-cE_0)+O\left(\frac{v^2}{c^2}\right). \tag{95.28}$$

Regrouping terms and using (95.22) we find

$$\begin{aligned}\frac{1}{c^2}\frac{DF_0}{Dt}+\frac{\partial P_0}{\partial z}+\frac{2}{c^2}\frac{\partial v}{\partial z}F_0&=-\frac{\kappa_0 F_0}{c}+\frac{v}{c^2}\left[\kappa_0(4\pi B_0-cE_0)-\frac{\partial F_0}{\partial z}\right]\\&=-\frac{\kappa_0 F_0}{c}+O\left(\frac{v^2}{c^2}\right),\end{aligned} \tag{95.29}$$

which is identical to (95.23).

NONINERTIAL FRAME FORMULATION

Following Lindquist (**L5**) and Castor (**C3**), we now derive the comoving-frame transfer equation directly in a noninertial Lagrangean frame. Again for convenience we use units in which $h = c = 1$, converting to physical units at a later stage. The photon Boltzmann equation in a noninertial frame is (cf. §92)

$$M^\alpha(\partial\mathcal{I}/\partial x^\alpha) + \dot{M}^\alpha(\partial\mathcal{I}/\partial M^\alpha) = e - a\mathcal{I} = (\delta\mathcal{I}/\delta\ell)_{\text{coll}} \qquad (95.30)$$

where $\mathcal{I} \equiv I/\nu^3$, $e \equiv \eta_\nu/\nu^2$, and $a \equiv \nu\chi_\nu$. Furthermore, $\dot{M}^\alpha \equiv (dM^\alpha/d\ell)$ where ℓ is an affine path-length parameter chosen to satisfy (92.3).

Photon trajectories are *geodesics* in the curved spacetime of the comoving frame. Therefore, the intrinsic derivative $(\delta M^\alpha/\delta\ell)$ is identically zero along a photon trajectory, and from equation (A3.100) we have

$$(\delta M^\alpha/\delta\ell) = (dM^\alpha/d\ell) + \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} M^\beta(dx^\gamma/d\ell) \equiv 0, \qquad (95.31)$$

or, in light of (92.3),

$$\dot{M}^\alpha = (dM^\alpha/d\ell) = -\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} M^\beta M^\gamma. \qquad (95.32)$$

Hence we can rewrite (95.30) as

$$M^\alpha(D\mathcal{I}/Dx^\alpha) = e - a\mathcal{I} = (\delta\mathcal{I}/\delta\ell)_{\text{coll}} \qquad (95.33)$$

where the operator

$$(D/Dx^\alpha) \equiv (\partial/\partial x^\alpha) - \begin{Bmatrix} \gamma \\ \alpha\beta \end{Bmatrix} M^\beta(\partial/\partial M^\gamma). \qquad (95.34)$$

The Christoffel symbols in (95.34) are to be derived from the spacetime metric, and in general will not vanish in the (noninertial) comoving frame even in Cartesian coordinates. We must now recast (95.33) and (95.34) into a more useful form.

In writing (95.33) we have tacitly assumed that the invariant intensity is defined for all possible four-momenta. But M^α is a null vector, hence $\mathcal{I}(x^\alpha, M^\alpha)$ is actually defined only for those arguments M^α that lie on the null cone. We must therefore calculate (D/Dx^α) in such a way as to assure that M^α remains on the null cone as a photon propagates. One way of proceeding is to treat the contravariant space components M^i as independent coordinates, and to calculate (D/Dx^α) as an operator for the subset of vectors $\mathbf{M}$ of constant (null) length. But this approach is cumbersome, especially for systems having special symmetries (e.g., spherical symmetry) where simplifications are often possible. For such systems it is much more convenient to work in an *orthonormal* coordinate frame, using variables adapted to the symmetries in the problem.

Thus let M^α denote the contravariant components of $\mathbf{M}$ with respect to some coordinate system x^α that has a general metric $g_{\alpha\beta}$. Then

$$\mathbf{M} = M^\alpha \boldsymbol{\varepsilon}_\alpha \qquad (95.35)$$

where the $\boldsymbol{\varepsilon}_\alpha$ are basis vectors of the coordinate system. In the neighborhood of any point $\mathbf{x}$, introduce an orthonormal *tetrad frame* $\boldsymbol{\varepsilon}_a(\mathbf{x})$, $(a = 0, 1, 2, 3)$, such that

$$\boldsymbol{\varepsilon}_a(\mathbf{x}) \cdot \boldsymbol{\varepsilon}_b(\mathbf{x}) = \eta_{ab} \tag{95.36}$$

where η_{ab} is the Lorentz metric. Relative to this frame, we can express $\mathbf{M}$ in terms of its *tetrad components* M^a as

$$\mathbf{M} = M^a \boldsymbol{\varepsilon}_a. \tag{95.37}$$

Write the transformation between the two coordinate systems as

$$\boldsymbol{\varepsilon}_a = \varepsilon^\alpha_a \boldsymbol{\varepsilon}_\alpha \tag{95.38a}$$

and

$$\boldsymbol{\varepsilon}_\alpha = \varepsilon^a_\alpha \boldsymbol{\varepsilon}_a. \tag{95.38b}$$

Then clearly

$$M^a = \varepsilon^a_\alpha M^\alpha \tag{95.39a}$$

and

$$M^\alpha = \varepsilon^\alpha_a M^a. \tag{95.39b}$$

Now suppose we choose the particular coordinate transformation $x^\alpha \to x'^\alpha \equiv x^\alpha$ and $M^\alpha \to M^a = \varepsilon^a_\alpha(\mathbf{x}) M^\alpha$ that leaves the coordinate system unchanged, but expresses the photon momentum in terms of tetrad components. Then if we regard $\mathscr{I}$ as a function of (x^α, M^a), the transfer equation can be written

$$\begin{aligned} M^a(D\mathscr{I}/Dx^a) &= (\delta\mathscr{I}/\delta\ell)_{\rm coll} = (\partial\mathscr{I}/\partial x^\alpha)(dx^\alpha/d\ell) + (\partial\mathscr{I}/\partial M^b)(dM^b/d\ell) \\ &= M^a\varepsilon^\alpha_a(\partial\mathscr{I}/\partial x^\alpha) + (\partial\mathscr{I}/\partial M_b)(dM^b/d\ell). \end{aligned} \tag{95.40}$$

The operator $\partial_a \equiv \varepsilon^\alpha_a(\partial/\partial x^\alpha)$ is known as the *Pfaffian derivative.*

To calculate $(dM^b/d\ell)$, we recall that photon trajectories are geodesics in the original coordinate system, hence

$$\begin{aligned} \frac{\delta M^\beta}{\delta\ell} &= M^\beta_{,\alpha}\frac{dx^\alpha}{dl} + \begin{Bmatrix}\beta\\ \alpha\gamma\end{Bmatrix} M^\gamma \frac{dx^\alpha}{d\ell} = (\varepsilon^\beta_c M^c)_{,\alpha}\frac{dx^\alpha}{d\ell} + \begin{Bmatrix}\beta\\ \alpha\gamma\end{Bmatrix} M^\alpha M^\gamma \\ &= \varepsilon^\beta_c(dM^c/d\ell) + M^c\varepsilon^\beta_{c,\alpha}M^\alpha + \begin{Bmatrix}\beta\\ \alpha\gamma\end{Bmatrix} M^\alpha M^\gamma \equiv 0. \end{aligned} \tag{95.41}$$

Therefore

$$\begin{aligned} \varepsilon^b_\beta\varepsilon^\beta_c\frac{dM^c}{d\ell} = \delta^b_c\frac{dM^c}{d\ell} = \frac{dM^b}{d\ell} &= -\varepsilon^\alpha_a\varepsilon^b_\beta\left(\varepsilon^\beta_{c,\alpha} + \begin{Bmatrix}\beta\\ \alpha\gamma\end{Bmatrix}\varepsilon^\gamma_c\right)M^aM^c \\ &= -\varepsilon^\alpha_a\varepsilon^b_\beta\varepsilon^\beta_{c;\alpha}M^aM^c. \end{aligned} \tag{95.42}$$

Then defining the *Ricci rotation coefficient* to be

$$\Gamma^b_{ac} \equiv \varepsilon^\alpha_a\varepsilon^b_\beta\varepsilon^\beta_{c;\alpha}, \tag{95.43}$$

the transfer equation becomes

$$M^a(D\mathscr{I}/Dx^a) = M^a[\partial_a - \Gamma^b_{ac}M^c(\partial/\partial M^b)]\mathscr{I} = e - a\mathscr{I}. \tag{95.44}$$

Notice that, unlike Christoffel symbols, the rotation coefficients are not symmetric in the two lower indices.

Because **M** is a null vector, only three of its tetrad components can be independent, as any three suffice to determine the fourth. Therefore in the evaluation of (95.44) we need differentiate only with three components of **M**, which we take to be the three space components M^a, $(a = 1, 2, 3)$.

We now specialize (95.44) to spherical symmetry. Choose a comoving-frame metric of the general form

$$ds^2 = -e^{2\Psi}\,d\tau^2 + e^{2\Lambda}\,d\mathfrak{r}^2 + R^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{95.45}$$

where $\mathfrak{r}$ is a generalized Lagrangean radial coordinate, and Ψ, Λ, and R are functions of $\mathfrak{r}$ and τ only. In spherical symmetry the derivatives $(\partial/\partial\theta)$ and $(\partial/\partial\phi)$ are identically zero, so we need calculate only terms containing $(\partial/\partial\tau)$ and $(\partial/\partial\mathfrak{r})$. From straightforward calculation one finds that the nonzero Christoffel symbols for (95.45) are:

$$\left.\begin{aligned}
&\begin{Bmatrix}0\\00\end{Bmatrix} = (\partial\Psi/\partial\tau), \qquad \begin{Bmatrix}0\\11\end{Bmatrix} = \exp[2(\Lambda-\Psi)](\partial\Lambda/\partial\tau),\\
&\begin{Bmatrix}0\\22\end{Bmatrix} = \exp(-2\Psi)R(\partial R/\partial\tau),\\
&\begin{Bmatrix}0\\33\end{Bmatrix} = \exp(-2\Psi)R(\partial R/\partial\tau)\sin^2\theta, \qquad \begin{Bmatrix}0\\10\end{Bmatrix} = (\partial\Psi/\partial\mathfrak{r}),\\
&\begin{Bmatrix}1\\00\end{Bmatrix} = \exp[2(\Psi-\Lambda)](\partial\Psi/\partial\mathfrak{r}), \qquad \begin{Bmatrix}1\\11\end{Bmatrix} = (\partial\Lambda/\partial\mathfrak{r}),\\
&\begin{Bmatrix}1\\22\end{Bmatrix} = -\exp(-2\Lambda)R(\partial R/\partial\mathfrak{r}),\\
&\begin{Bmatrix}1\\33\end{Bmatrix} = -\exp(-2\Lambda)R(\partial R/\partial\mathfrak{r})\sin^2\theta,\\
&\begin{Bmatrix}1\\01\end{Bmatrix} = (\partial\Lambda/\partial\tau), \qquad \begin{Bmatrix}2\\33\end{Bmatrix} = -\sin\theta\cos\theta, \qquad \begin{Bmatrix}2\\02\end{Bmatrix} = R^{-1}(\partial R/\partial\tau),\\
&\begin{Bmatrix}2\\12\end{Bmatrix} = R^{-1}(\partial R/\partial\mathfrak{r}), \qquad \begin{Bmatrix}3\\03\end{Bmatrix} = R^{-1}(\partial R/\partial\tau), \qquad \begin{Bmatrix}3\\13\end{Bmatrix} = R^{-1}(\partial R/\partial\mathfrak{r}),\\
&\text{and}\\
&\begin{Bmatrix}3\\23\end{Bmatrix} = \cot\theta
\end{aligned}\right\} \tag{95.46}$$

At the event $(\tau, \mathfrak{r}, \theta, \phi)$ introduce the orthonormal basis

$$\boldsymbol{\varepsilon}_0 = e^{-\Psi}\boldsymbol{\varepsilon}_\tau, \qquad \boldsymbol{\varepsilon}_1 = e^{-\Lambda}\boldsymbol{\varepsilon}_{\mathfrak{r}}, \qquad \boldsymbol{\varepsilon}_2 = R^{-1}\boldsymbol{\varepsilon}_\theta, \qquad \text{and} \qquad \boldsymbol{\varepsilon}_3 = (R\sin\theta)^{-1}\boldsymbol{\varepsilon}_\phi. \tag{95.47}$$

One then sees that the transformation matrix ε_a^α is diagonal:

$$\varepsilon_a^\alpha = \begin{pmatrix} e^{-\Psi} & 0 & 0 & 0 \\ 0 & e^{-\Lambda} & 0 & 0 \\ 0 & 0 & R^{-1} & 0 \\ 0 & 0 & 0 & (R\sin\theta)^{-1} \end{pmatrix}, \tag{95.48}$$

whence we have

$$\varepsilon_\alpha^a \equiv (\varepsilon_a^\alpha)^{-1} = \begin{pmatrix} e^{\Psi} & 0 & 0 & 0 \\ 0 & e^{\Lambda} & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R\sin\theta \end{pmatrix}, \tag{95.49}$$

Furthermore, write $\mathbf{M}$ in terms of spherical coordinates with $\boldsymbol{\varepsilon}_1$ taken to be the polar axis:

$$M^0 = \nu, \qquad M^1 = \nu\cos\Theta, \qquad M^2 = \nu\sin\Theta\cos\Phi, \qquad M^3 = \nu\sin\Theta\sin\Phi, \tag{95.50}$$

where ν is the photon's energy. We can then compute the Jacobian $J(M^1, M^2, M^3/\nu, \Theta, \Phi)$ and its inverse

$$J^{-1} = \frac{\partial(\nu, \Theta, \Phi)}{\partial(M^1, M^2, M^3)}$$

$$= \begin{pmatrix} \cos\Theta & -(\sin\Theta)/\nu & 0 \\ \sin\Theta\cos\Phi & (\cos\Theta\cos\Phi)/\nu & -(\sin\Phi)/\nu\sin\Theta \\ \sin\Theta\sin\Phi & (\cos\Theta\sin\Phi)/\nu & (\cos\Phi)/\nu\sin\Theta \end{pmatrix}, \tag{95.51}$$

whence we have

$$(\partial/\partial M^1) = \mu(\partial/\partial\nu) + \nu^{-1}(1-\mu^2)(\partial/\partial\mu), \tag{95.52}$$

$$(\partial/\partial M^2) = (1-\mu^2)^{1/2}\cos\Phi[(\partial/\partial\nu) - \nu^{-1}\mu(\partial/\partial\mu)], \tag{95.53}$$

and

$$(\partial/\partial M^3) = (1-\mu^2)^{1/2}\sin\Phi[(\partial/\partial\nu) - \nu^{-1}\mu(\partial/\partial\mu)], \tag{95.54}$$

where $\mu \equiv \cos\Theta$. Here we have dropped $(\partial/\partial\Phi)$ because of azimuthal symmetry.

We now must compute the Ricci rotation coefficients. Because ε_a^α and ε_α^a are diagonal, (95.43) reduces to

$$\Gamma^b_{ac} = \varepsilon_a^a(\varepsilon^{-1})^b_b\left(\varepsilon_c^c\left\{\begin{matrix} b \\ ac \end{matrix}\right\} + \varepsilon^b_{c,a}\,\delta^b_c\right), \tag{95.55}$$

where there is no sum on repeated indices. We can ignore terms with $b = 0$ because in (95.44) we differentiate only with respect to space components of M^b. Following (**L5**), define the operators

$$D_\tau \equiv e^{-\Psi}(\partial/\partial\tau) \quad (95.56a)$$

and

$$D_\imath \equiv e^{-\Lambda}(\partial/\partial\imath), \quad (95.56b)$$

and the auxiliary variables

$$U \equiv D_\tau R \quad (95.57a)$$

and

$$\Gamma \equiv D_\imath R. \quad (95.57b)$$

Using (95.46), (95.48), and (95.49) in (95.55) we find that the nonzero Ricci coefficients are

$$\left.\begin{aligned} &\Gamma^1_{00} = D_\imath\Psi, \qquad \Gamma^1_{22} = \Gamma^1_{33} = -\Gamma/R, \qquad \Gamma^1_{10} = D_\tau\Lambda,\\ &\Gamma^2_{33} = -R^{-1}\cot\theta, \qquad \Gamma^2_{20} = \Gamma^3_{30} = U/R,\\ &\Gamma^2_{21} = \Gamma^3_{31} = \Gamma/R, \qquad \text{and} \qquad \Gamma^3_{32} = R^{-1}\cot\theta. \end{aligned}\right\} \quad (95.58)$$

In the transfer equation (95.44) we then have

$$M^a\partial_a = M^a\varepsilon^\alpha_a(\partial/\partial x^\alpha) = \nu D_\tau + \mu\nu D_\imath, \quad (95.59)$$

while

$$\begin{aligned} M^aM^c\Gamma^b_{ac}(\partial/\partial M^b) &= (M^0M^0\Gamma^1_{00} + M^2M^2\Gamma^1_{22} + M^3M^3\Gamma^1_{33} + M^0M^1\Gamma^1_{10})(\partial/\partial M^1)\\ &\quad + (M^3M^3\Gamma^2_{33} + M^1M^2\Gamma^2_{21} + M^0M^2\Gamma^2_{20})(\partial/\partial M^2) \quad (95.60)\\ &\quad + (M^1M^3\Gamma^3_{31} + M^0M^3\Gamma^3_{30} + M^2M^3\Gamma^3_{32})(\partial/\partial M^3). \end{aligned}$$

Substituting (95.50) and (95.52) to (95.54) into (95.60), collecting terms, and using the results along with (95.59) in (95.44) we obtain finally the comoving-frame transfer equation

$$\begin{aligned} &D_\tau\mathscr{I} + \mu D_\imath\mathscr{I} - \nu[\mu D_\imath\Psi + \mu^2 D_\tau\Lambda + (1-\mu^2)(U/R)](\partial\mathscr{I}/\partial\nu)\\ &+ (1-\mu^2)\{(\Gamma/R) - D_\imath\Psi + \mu[(U/R) - D_\tau\Lambda]\}(\partial\mathscr{I}/\partial\mu) = \nu^{-1}(e - a\mathscr{I}). \end{aligned} \quad (95.61)$$

Equation (95.61) is exact for the general metric (95.45). To apply it to a particular flow we must obtain explicit expressions for the coefficients in the metric; it is at this point that we must forsake exactness if we wish to obtain analytical results. One sees that some kind of approximation must be made by realizing that in general the acceleration field $\mathbf{a}(\mathbf{r}, t)$ can be arbitrarily complicated, and by recalling that the principle of equivalence implies that this field can be viewed as resulting from the gravitational field of an arbitrarily complex distribution of masses. Thus an attempt to obtain an exact analytical metric for an arbitrary flow field is as difficult as solving exactly the field equations of general relativity for an arbitrary mass distribution, which is not possible by known methods. In practice, it is feasible to work analytically only to $O(v/c)$. An alternative is to construct the metric numerically; but by doing so we forsake having explicit analytical expressions for the metric and the transfer equation. See (**G1**) for a discussion of the numerical approach in the context of radiative transfer.

For one-dimensional spherically symmetric flows, Castor (**C3**) adopted inertial-frame coordinates (t', r, θ, ϕ) and Lagrangean coordinates (t, M_r, θ, ϕ), and related them by the coordinate transformation

$$M_r(r, t') = \int_0^r 4\pi(r')^2\rho(r', t')\,dr' \tag{95.62}$$

and

$$t(r, t') = t' - c^{-2}\int_0^r v(r', t')\,dr', \tag{95.63}$$

where $v = (\partial r/\partial t') = -(4\pi r^2\rho)^{-1}(\partial M_r/\partial t')_r$ is the fluid velocity, Equations (95.62) and (95.63) provide an $O(v/c)$ approximation to a local Lorentz transformation between the inertial and comoving frames in the neighborhood of the event (r, t'). From these equations one readily finds

$$dx \equiv (dM_r/4\pi r^2\rho) = dr - v\,dt' \tag{95.64}$$

and

$$dt = (1 - I/c^2)\,dt' - (v/c^2)\,dr, \tag{95.65}$$

where

$$I \equiv \int_0^r [\partial v(r', t')/\partial t']\,dr'. \tag{95.66}$$

Solving for dr and dt' we have

$$dr = [(1 - I/c^2)/D]\,dx + (v/D)\,dt \tag{95.67}$$

and

$$dt' = (v/c^2 D)\,dx + D^{-1}\,dt \tag{95.68}$$

where

$$D \equiv 1 - (I + v^2)/c^2. \tag{95.69}$$

Substituting (95.67) and (95.68) into the inertial-frame metric

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) - c^2(dt')^2 \tag{95.70}$$

we obtain the comoving-frame metric

$$ds^2 = F(dM_r/4\pi r^2\rho)^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) - G\,dt^2 - 2H\,dM_r\,dt \tag{95.71}$$

where [see (**M6**)]

$$F = [(1 - I/c^2)^2 - (v^2/c^2)]/D^2, \tag{95.72}$$

$$G = (c^2 - v^2)/D^2, \tag{95.73}$$

and

$$H = vI/(4\pi r^2\rho c^2 D^2). \tag{95.74}$$

Inasmuch as we are interested in final results correct to $O(v/c)$, we may now discard terms of $O(v^2/c^2)$. We see by inspection that H is $O(v^2/c^2)$, and hence can be dropped, while $F = 1 + O(v^2/c^2)$. For G we have

$$G = c^2/(1 - 2I/c^2) + O(v^2/c^2) = c^2 + 2I + O(v^2/c^2); \tag{95.75}$$

I/c^2 can be $O(v/c)$ for radiation-flow time scales $t_R \sim \Delta r/c$ or when the fluid acceleration is comparable to (cv/r) or $c(dv/dr)$, and hence should be retained.

Comparing (95.71) with (95.45) in which $d\tau \equiv dt$ and $d\mathfrak{z} \equiv dM_r$ we can make the identifications

$$R \equiv r, \qquad \Lambda \equiv -\ln(4\pi r^2 \rho), \qquad \text{and} \qquad \Psi \equiv \tfrac{1}{2}\ln(c^2+2I), \quad (95.76)$$

whence we find, to $O(v/c)$,

$$D_\tau \equiv c^{-1}(\partial/\partial t) = c^{-1}(D/Dt) \qquad \text{and} \qquad D_{\mathfrak{z}} \equiv (4\pi r^2\rho)(\partial/\partial M_r) \equiv (\partial/\partial r). \quad (95.77)$$

Here we noted that the time derivative calculated in the comoving frame is identical to the customary Lagrangean (D/Dt). From (95.57) and (95.77) we find $U = (v/c)$, $\Gamma \equiv 1$,

$$D_{\mathfrak{z}}\Psi = \frac{1}{c^2+2I}\frac{\partial}{\partial r}\left(\int_0^r \frac{\partial v}{\partial t'}\,dr'\right) = \frac{1}{c^2}\frac{\partial v}{\partial t'} + O\left(\frac{v^2}{c^2}\right) = \frac{a}{c^2}, \quad (95.78)$$

and

$$D_\tau \Lambda = -c^{-1}[D(\ln\rho)/Dt + (2v/r)]. \quad (95.79)$$

Using (95.76) to (95.79) in (95.61) and expressing $\mathcal{I}$, α, and ε in terms of $I_0(\mu_0, \nu_0)$, $\chi_0(\nu_0)$, and $\eta_0(\nu_0)$, we find, after some elementary reductions, the comoving-frame transfer equation

$$\begin{aligned}
&\frac{1}{c}\frac{DI_0(\mu_0,\nu_0)}{Dt} + 4\pi\rho\mu_0\frac{\partial}{\partial M_r}[r^2 I_0(\mu_0,\nu_0)] \\
&\quad + \frac{\partial}{\partial\mu_0}\left\{(1-\mu_0^2)\left[\frac{1}{r} + \frac{\mu_0}{c}\left(\frac{3v}{r} + \frac{D\ln\rho}{Dt}\right) - \frac{a}{c^2}\right] I_0(\mu_0,\nu_0)\right\} \\
&\quad - \frac{\partial}{\partial\nu_0}\left\{\nu_0\left[(1-3\mu_0^2)\frac{v}{cr} - \frac{\mu_0^2}{c}\frac{D\ln\rho}{Dt} + \frac{\mu_0 a}{c^2}\right] I_0(\mu_0,\nu_0)\right\} \\
&\quad + \left[(1-3\mu_0^2)\frac{v}{cr} - \frac{(1+\mu_0^2)}{c}\frac{D\ln\rho}{Dt} + \frac{2\mu_0 a}{c^2}\right] I_0(\mu_0,\nu_0) \\
&= \eta_0(\nu_0) - \chi(\nu_0) I_0(\mu_0,\nu_0).
\end{aligned} \quad (95.80)$$

This equation is fully Lagrangean in the sense that *all* radiation and material properties are in the comoving frame, the independent variable M_r is Lagrangean, and the time derivatives (D/Dt) are evaluated in a moving fluid element. Recalling the equation of continuity

$$(D\ln\rho/Dt) = -r^{-2}[\partial(r^2 v)/\partial r] = -(\partial v/\partial r) - (2v/r), \quad (95.81)$$

one easily sees that (95.80) is identical to (95.17). We thus have two logically independent derivations of the result.

Taking angular moments of (95.80) we obtain the monochromatic radiation energy equation

$$\begin{aligned}\frac{DE_0(\nu_0)}{Dt}&+4\pi\rho_0\frac{\partial}{\partial M_r}[r^2F_0(\nu_0)]-\frac{v}{r}[3P_0(\nu_0)-E_0(\nu_0)]\\&-\frac{D\ln\rho}{Dt}[E_0(\nu_0)+P_0(\nu_0)]+\frac{2a}{c^2}F_0(\nu_0)\\&+\frac{\partial}{\partial\nu_0}\left[\!\!\left[\nu_0\left\{\frac{v}{r}[3P_0(\nu_0)-E_0(\nu_0)]+\frac{D\ln\rho}{Dt}P_0(\nu_0)-\frac{a}{c^2}F_0(\nu_0)\right\}\right]\!\!\right]\\&=4\pi\eta_0(\nu_0)-c\chi_0(\nu_0)E_0(\nu_0),\end{aligned} \tag{95.82}$$

and the monochromatic radiation momentum equation

$$\begin{aligned}\frac{1}{c^2}\frac{DF_0(\nu_0)}{Dt}&+4\pi r^2\rho\frac{\partial P_0(\nu_0)}{\partial M_r}+\frac{3P_0(\nu_0)-E_0(\nu_0)}{r}\\&-\frac{2}{c^2}\left(\frac{v}{r}+\frac{D\ln\rho}{Dt}\right)F_0(\nu_0)+\frac{a}{c^2}[E_0(\nu_0)+P_0(\nu_0)]\\&+\frac{\partial}{\partial\nu_0}\left[\!\!\left[\nu_0\left\{\frac{v}{c^2r}[3Q_0(\nu_0)-F_0(\nu_0)]+\frac{1}{c^2}\frac{D\ln\rho}{Dt}Q_0(\nu_0)-\frac{a}{c^2}P_0(\nu_0)\right\}\right]\!\!\right]\\&=-\frac{\chi_0(\nu_0)}{c}F_0(\nu_0),\end{aligned} \tag{95.83}$$

which are equivalent to (95.18) and (95.19).

Integrating over frequency we obtain the radiation energy equation

$$\begin{aligned}\frac{DE_0}{Dt}+4\pi\rho\frac{\partial(r^2F_0)}{\partial M_r}-\frac{v}{r}(3P_0-E_0)-\frac{D\ln\rho}{Dt}(E_0+P_0)+\frac{2aF_0}{c^2}\\=\int_0^\infty[4\pi\eta_0(\nu_0)-c\chi_0(\nu_0)E_0(\nu_0)]\,d\nu_0\end{aligned} \tag{95.84}$$

and the radiation momentum equation

$$\begin{aligned}\frac{1}{c^2}\frac{DF_0}{Dt}+4\pi r^2\rho\frac{\partial P_0}{\partial M_r}+\frac{3P_0-E_0}{r}-\frac{2}{c^2}\left(\frac{v}{r}+\frac{D\ln\rho}{Dt}\right)F_0+\frac{a}{c^2}(E_0+P_0)\\=-\frac{1}{c}\int_0^\infty\chi_0(\nu_0)F_0(\nu_0)\,d\nu_0,\end{aligned} \tag{95.85}$$

which are equivalent to (95.20) and (95.21). These equations also follow directly from

$$R_{0;\beta}^{\alpha\beta}=-G_0^\alpha \tag{95.86}$$

where $R_0^{\alpha\beta}$ is given by (91.9) with all radiation quantities evaluated in the comoving frame, G_0^α is given by (91.25), and the covariant derivatives are evaluated in the curved spacetime of the fluid frame. Using (A3.89) with

Christoffel symbols calculated in the metric (95.71), one can show that (95.86) does, in fact, yield (95.84) and (95.85).

Equations (95.84) and (95.85) apply in spherical symmetry. Buchler has shown (**B2**) that tensorial forms of these equations, applicable in any geometry, are

$$\rho \frac{D}{Dt}\left(\frac{E_0}{\rho}\right) + \nabla \cdot \mathbf{F}_0 + \mathsf{P}_0 : \nabla \mathbf{v} + \frac{2}{c^2}\mathbf{a} \cdot \mathbf{F}_0 + cG_0^0 = 0, \qquad (95.87)$$

and

$$\frac{\rho}{c^2}\frac{D}{Dt}\left(\frac{\mathbf{F}_0}{\rho}\right) + \nabla \cdot \mathsf{P}_0 + \frac{1}{c^2}\mathbf{F}_0 \cdot \nabla \mathbf{v} + \frac{1}{c^2}(E_0\mathbf{a} + \mathbf{a} \cdot \mathsf{P}_0) + \mathbf{G}_0 = 0. \quad (95.88)$$

The term $\mathsf{P}_0 : \nabla \mathbf{v}$ in (95.87) is dyadic notation for the contraction of P_0 with $\nabla \mathbf{v}$. Buchler also gives tensorial forms for the monochromatic moment equations [see his equations (9) and (10)].

IMPORTANCE OF $O(v/c)$ TERMS

In §93 we showed that in order to solve correctly the inertial-frame transfer equation and its moments one must retain terms that are formally $O(v/c)$ (cf. §93). Building on the discussion by Castor (**C3**), we now show that the same conclusion applies to the comoving-frame radiation and momentum equations. In making estimates of the relative sizes of terms we shall ignore the acceleration terms [which are never larger than $O(v/c)$], and consider $(\partial v/\partial r)$, (v/r), and $(D \ln \rho/Dt)$ to be $O(v/l)$. In the diffusion regime, we shall use results to be derived in §97 for estimating the sizes of the net absorption-emission terms, $\mathbf{F}_0$, and $(3P_0 - E_0)$.

Consider first the radiation energy equation (95.84); group the net absorption-emission into a single term. In the streaming limit, dimensional analysis suggests that on a fluid-flow time-scale the five terms in (95.84) scale as $(v/c):1:(v/c):(v/c):(l/\lambda_p)$, hence we need retain only the flux divergence and the absorption-emission terms; the radiation field is quasi-static. On a radiation-flow time scale we must also retain the (D/Dt) term. If the material is essentially in radiative equilibrium, the absorption-emission terms cancel almost exactly, and the (D/Dt) and velocity-dependent terms, although small, may significantly affect the energy balance; we should then retain all terms. In the static diffusion limit, the terms scale as $(v/c)(l/\lambda_p):1:(v/c)^2:(v/c)(l/\lambda_p):1$, hence only the flux-divergence and absorption-emission terms need be retained. As $(v/c) \to (\lambda_p/l)$, all terms except the one containing $(3P_0 - E_0)$ are of the same order, and all must be kept. In the dynamic diffusion regime the scaling is $1:(c/v)(\lambda_p/l):(v/c)(\lambda_p/l):1:1$. The dominant terms are the rate of change of the energy density, the rate of work done by radiation pressure, and the net absorption-emission terms; the flux divergence is of less importance than in other regimes, and again we can drop $(3P_0 - E_0)$. In summary, *to guarantee the correct radiation energy balance in all regimes, we must retain all terms in* (95.84) *except the acceleration term.*

Now consider the radiation momentum equation (95.85). In the streaming limit, dimensional analysis suggests that on a fluid-flow time scale the terms scale as $(v/c):1:1:(v/c):(l/\lambda_p)$. Hence we need retain only $\nabla \cdot \mathsf{P}_0$ and the integral of $\chi_0 F_0/c$. If we follow radiation flow, the (D/Dt) term must also be kept. In the diffusion regime the terms scale as $(v/c)(\lambda_p/l):1:(v/c)(\lambda_p/l):(v/c)(\lambda_p/l):1$, hence we can drop (D/Dt), $(3P_0-E_0)$, and the velocity-dependent terms. This result contrasts strongly with that for the inertial-frame radiation momentum equation (where it is essential to retain all the velocity-dependent terms to obtain the correct inertial-frame flux), and reveals an important advantage of the Lagrangean formulation. *In summary, in solving the comoving-frame radiation momentum equation* (95.85) *on a fluid-flow time scale we can drop the time derivative and all velocity-dependent terms.*

Castor (**C3**) arrives at the same conclusions for a pulsating star where (D/Dt) is of the order of ω, the pulsation frequency.

96. Comoving-Frame Equations of Radiation Hydrodynamics

We are now in a position to write the Lagrangean equations of radiation hydrodynamics. We consider one-dimensional spherically symmetric flows; the corresponding planar equations are obtained by taking the limit $(1/r)\to 0$. We ignore the acceleration terms in the radiation energy and momentum equations, which are $O(v^2/c^2)$ on fluid-flow time scales (but see §97).

THE MOMENTUM EQUATION

The simplest way to obtain the comoving-frame momentum equation is to reduce the relativistically correct equation (94.12a) to the proper frame, in which $\mathbf{v}=0$ instantaneously. We then have, to $O(v/c)$,

$$\rho_{000}(D\mathbf{v}/Dt)=\mathbf{f}-\nabla p+\mathbf{G}_0. \qquad (96.1)$$

For nonrelativistic fluids $(p+\rho_0 e)\ll \rho_0 c^2$, and we can ignore the difference between ρ_{000} and ρ. Specializing (96.1) to one-dimensional spherically symmetric flow we find

$$\rho(Dv/Dt)=-(GM_r\rho/r^2)-(\partial p/\partial r)+(1/c)\int_0^\infty \chi_0(\nu_0)F_0(\nu_0)\,d\nu_0, \qquad (96.2)$$

which states that a fluid element accelerates in response to applied external forces (e.g., gravity), the pressure gradient, and the force exerted by the radiation on the material as measured in its rest frame. The velocity-dependent terms in the inertial-frame momentum equation vanish in the Lagrangean frame.

To obtain the comoving-frame analogue of (94.12b), we use (95.85) to eliminate the integral in (96.2), which yields

$$\rho\frac{Dv}{Dt}+\frac{1}{c^2}\frac{DF_0}{Dt}=-\frac{GM_r\rho}{r^2}-\left(\frac{\partial p}{\partial r}+\frac{\partial P_0}{\partial r}+\frac{3P_0-E_0}{r}\right)+\frac{2}{c^2}\left(\frac{v}{r}+\frac{D\ln\rho}{Dt}\right)F_0. \tag{96.3}$$

We can also derive (96.3) by evaluating (94.12b) directly in the comoving frame provided that we replace $[c^{-2}(\partial\mathbf{F}/\partial t)+\nabla\cdot\mathsf{P}]$ with $(R_0^{1\beta})_{;\beta}$ and calculate the covariant derivative using the (nonzero) Christoffel symbols obtained from the metric (95.71). Regrouping terms in (96.3), we can write it in the more instructive form

$$\rho\frac{D}{Dt}\left[v+\left(\frac{F_0}{c^2\rho}\right)\right]=-\frac{GM_r\rho}{r^2}-\frac{\partial(p+P_0)}{\partial r}-\frac{3P_0-E_0}{r}-\frac{1}{c^2}\left(\frac{\partial v}{\partial r}\right)F_0, \tag{96.4}$$

which states that the rate of change of the total (material plus radiative) momentum density in a radiating fluid equals the applied force minus the divergence of the total stress, minus an additional (relativistic) term that arises because the radiant energy flux has inertia (cf. §97).

On a fluid-flow time scale both terms containing F_0 in (96.4) are $O(v/c)$ in the streaming limit, and $O(\lambda_p v/lc)$ in the diffusion limit, relative to $(\partial P_0/\partial r)$, and can be dropped in practical calculations. Hence another useful form of the Lagrangean momentum equation is

$$\rho(D\mathbf{v}/Dt)=\mathbf{f}-\nabla p-\nabla\cdot\mathsf{P}_0. \tag{96.5}$$

Equation (96.5) is slightly more approximate than (96.2), but assumes a particularly simple form in the diffusion limit, where $\nabla\cdot\mathsf{P}_0$ reduces to ∇P_0, so that the fluid acceleration depends on the total (gas plus radiation) pressure gradient.

THE GAS-ENERGY EQUATION

The comoving-frame gas-energy equation follows directly from the relativistic equation (94.18) by evaluating $V_\alpha F^\alpha$ and $V_\alpha G^\alpha$ in the proper frame. We obtain

$$\rho_0\{(De/D\tau)+p[D(1/\rho_0)/D\tau]\}=c(F_0^0+G_0^0), \tag{96.6}$$

where G_0^0 is given by (91.25a). For ordinary body forces $cF_0^0=(\mathbf{v}\cdot\mathbf{f})_0=0$. But in the presence of nonmechanical energy sources cF_0^0 equals the rate, per unit volume, of energy input to the material, as measured in the fluid frame (cf. §37). For example, in stellar interiors thermonuclear reactions irreversibly release ε ergs $g^{-1}\,s^{-1}$ into the material. In this case

$$\rho_0\left[\frac{De}{D\tau}+p\frac{D}{D\tau}\left(\frac{1}{\rho_0}\right)\right]=\int_0^\infty[c\chi_0(\nu_0)E_0(\nu_0)-4\pi\eta_0(\nu_0)]\,d\nu_0+\rho_0\varepsilon. \tag{96.7}$$

Equation (96.7) is the first law of thermodynamics for the material; it states that the rate of change of the material energy density plus the rate of work done by the material pressure equals the net rate of energy input from the radiation field and thermonuclear sources, all per unit mass. In what follows we work to $O(v/c)$, hence in (96.7) we replace ρ_0 by ρ and $(D/D\tau)$ by (D/Dt).

THE RADIATION-ENERGY EQUATION

By rearranging terms we can write (95.84) in a form that makes its physical content more apparent:

$$\rho\left[\frac{D}{Dt}\left(\frac{E_0}{\rho}\right)+P_0\frac{D}{Dt}\left(\frac{1}{\rho}\right)-(3P_0-E_0)\frac{v}{\rho r}\right] = \int_0^\infty [4\pi\eta_0(\nu_0)-c\chi_0(\nu_0)E_0(\nu_0)]\,d\nu_0-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_0). \tag{96.8}$$

The second and third terms on the left-hand side of (96.8) reduce to $P_0^{ij}v_{i;j}$, the contraction of the radiation-pressure and fluid-velocity tensors, hence equal the rate of work done by the radiation stress [cf. (27.7)]. Thus (96.8) is the first law of thermodynamics for the radiation field; it states that the rate of change of the radiation energy density, plus the rate of work done by radiation pressure, equals the net rate of energy input into the radiation field from the material, minus the net rate of radiant energy flow out of a fluid element by transport [again cf. (27.7)], all per unit mass.

THE FIRST LAW OF THERMODYNAMICS FOR THE RADIATING FLUID

Taking the sum of (96.7) and (96.8) we obtain the first law of thermodynamics for the radiating fluid:

$$\frac{D}{Dt}\left(e+\frac{E_0}{\rho}\right)+p\frac{D}{Dt}\left(\frac{1}{\rho}\right)+\left[P_0\frac{D}{Dt}\left(\frac{1}{\rho}\right)-(3P_0-E_0)\frac{v}{\rho r}\right]=\varepsilon-\frac{\partial}{\partial M_r}(4\pi r^2F_0), \tag{96.9}$$

which states that the rate of change of the total (material plus radiation) energy density in a fluid element plus the rate of work done by the total pressure in the element equals the rate of thermonuclear energy input into the element minus the rate of radiant energy loss by transport to adjacent fluid elements.

When the radiation field is isotropic (e.g., in the diffusion regime), (96.9) simplifies to

$$\frac{D}{Dt}\left(e+\frac{E_0}{\rho}\right)+(p+P_0)\frac{D}{Dt}\left(\frac{1}{\rho}\right)=\varepsilon-\frac{\partial L_r^0}{\partial M_r}, \tag{96.10}$$

where L_r^0 is the luminosity at radius r, measured in the comoving frame. In this limit, the radiating fluid behaves like a gas whose total energy density and pressure are the simple sums of the contributions from the radiation and material components.

In the equilibrium diffusion limit, (96.10) is the standard energy equation used in dynamical stellar evolution calculations [cf. (97.7)]. For a *static* medium, it reduces to one of the standard equations of stellar structure

$$(\partial L_r^0/\partial M_r) = \varepsilon, \qquad (96.11)$$

which apply to stable stars evolving on a *nuclear time scale* t_N, which is so long compared to dynamical times of interest (e.g., the free-fall time or a pulsation period) that the evolution is quasi-stationary and fluid motions can be neglected.

THE MECHANICAL ENERGY EQUATION

To obtain the fluid-frame mechanical energy equation we multiply the momentum equation (96.2) by v, which yields

$$\rho D(\tfrac{1}{2}v^2)/Dt = -(GM_r v\rho/r^2) - v(\partial p/\partial r) + (v/c)\int_0^\infty \chi_0(\nu_0)F_0(\nu_0)\,d\nu_0, \qquad (96.12)$$

which is identical to (24.8) if we lump the radiative force into f, and to (94.17a) except that here the radiation force is evaluated in the comoving frame.

THE TOTAL ENERGY EQUATION

To obtain a total energy equation we first rewrite (96.12) as

$$\frac{D}{Dt}\left(\tfrac{1}{2}v^2 - \frac{GM_r}{r}\right) + \frac{\partial}{\partial M_r}(4\pi r^2 v p) = p\frac{D}{Dt}\left(\frac{1}{\rho}\right) + \frac{v}{c\rho}\int_0^\infty \chi_0(\nu_0)F_0(\nu_0)\,d\nu_0. \qquad (96.13)$$

Next, substituting from (95.85) for the radiation force, and ignoring terms of $O(v^2/c^2)$ we obtain

$$\frac{D}{Dt}\left(\tfrac{1}{2}v^2 - \frac{GM_r}{r}\right) + \frac{\partial}{\partial M_r}[4\pi r^2 v(p+P_0)] = (p+P_0)\frac{D}{Dt}\left(\frac{1}{\rho}\right) - \frac{v}{\rho r}(3P_0 - E_0). \qquad (96.14)$$

Finally, adding (96.14) to (96.9) we have

$$\frac{D}{Dt}\left(e + \frac{E_0}{\rho} + \tfrac{1}{2}v^2 - \frac{GM_r}{r}\right) + \frac{\partial}{\partial M_r}\{4\pi r^2[v(p+P_0) + F_0]\} = \varepsilon, \qquad (96.15)$$

which is clearly a statement of overall energy conservation for the radiating fluid. All radiation quantities in (96.15) are to be evaluated in the comoving frame.

Equation (96.15) is essentially identical to equation (27.4), written in spherical coordinates, for an inviscid but conducting (via radiation) fluid whose internal energy density is the sum of the gas and radiation energy densities, and whose pressure equals the sum of the gas and radiation pressures, with the term on the right-hand side accounting for "external" energy input from thermonuclear reactions. This equation, with $4\pi r^2 F_0$ replaced by L_r, and E_0 and P_0 given their thermal equilibrium values, is the total energy equation used in dynamical stellar structure calculations [see, for example, (**C4,** eq. 6); (**F1,** eq. 3); (**K7,** eq. 15); or (**L2,** eq. 51.3)].

We can rewrite (96.15) in Eulerian coordinates as

$$\frac{\partial}{\partial t}\left(\rho e+E_0+\tfrac{1}{2}\rho v^2-\frac{GM_r\rho}{r}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left[\!\left[r^2\left\{\left[\rho\left(e+\tfrac{1}{2}v^2-\frac{GM_r}{r}\right)+p+P_0+E_0\right]v+F_0\right\}\right]\!\right]=\rho\varepsilon. \tag{96.16}$$

Then using (91.17a) and ignoring $O(v^2/c^2)$ terms in converting E_0 to E in the time derivative, we obtain

$$\frac{\partial}{\partial t}\left(\rho e+E+\tfrac{1}{2}\rho v^2-\frac{GM_r\rho}{r}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left[\!\left[r^2\left\{\left[\rho\left(e+\tfrac{1}{2}v^2-\frac{GM_r}{r}\right)+p\right]v+F\right\}\right]\!\right]=\rho\varepsilon, \tag{96.17}$$

which is identical to the Eulerian result (94.15b) when thermonuclear energy release is allowed. In (96.17), radiation quantities are now measured in the laboratory frame.

Assuming that $\dot{\mathcal{M}}$ is so small that we can neglect the time variation of M_r, we can write an explicit integral of (96.17) for the case of steady flow [cf. (24.22) for a nonradiating fluid]. We find

$$\dot{\mathcal{M}}[h+\tfrac{1}{2}v^2-(GM_r/r)]+L_r=4\pi\int_0^r \rho\varepsilon x^2\,dx. \tag{96.18}$$

That is, the total energy flux passing through a surface of radius r, consisting of the material energy flux (i.e., the mass flux times the enthalpy plus kinetic plus potential energy per unit mass) plus the luminosity radiated by the surface (measured in the lab frame) equals the total thermonuclear energy release in the volume bounded by the surface. In physical terms, (96.18) states that all the energy contained in radiation and in fluid motions in a star originates ultimately from thermonuclear energy release in the star's interior.

CONSISTENCY OF VARIOUS FORMS OF THE COMOVING-FRAME ENERGY AND MOMENTUM EQUATIONS

We now show that $O(v/c)$ terms must also be retained in order to obtain consistency among various forms of the comoving-frame energy equation, and between the comoving-frame and inertial-frame energy and momentum equations. Our discussion summarizes and extends a penetrating analysis of these issues by Castor (**C3**). An earlier, but incomplete, treatment was given by Wendroff (**W2**).

In an optically thin medium, or near a radiating surface of an opaque medium, the radiation field departs strongly from thermal equilibrium, hence J can differ markedly from B, the flux is large, and the radiation pressure tensor is anisotropic. In this regime, it is natural to describe the energy exchange between the material and radiation in terms of direct gains and losses, as in (96.7), and momentum exchange in terms of radiation forces acting on the material, as in (96.2).

In contrast, in the diffusion regime $J \rightarrow B$, so that the net absorption-emission term in (96.7) vanishes to high order, and the flux becomes a very small leak from the large reservoir of radiant energy. The radiation energy density and pressure both approach their equilibrium values, and the radiation pressure becomes isotropic. It is then natural to calculate the total energy content and pressure of the radiating fluid by adding the material and radiative contributions, and to use (96.9) as the energy equation and (96.5) as the momentum equation.

In any practical computation we must choose *one* form of the fluid energy equation even when the flow spans both the optically thin and thick limits. If the $O(v/c)$ terms are retained in the radiation energy equation (95.84), and this equation is solved simultaneously with either fluid energy equation, the choice is immaterial because exact consistency between the two is guaranteed. But suppose we *drop* the $O(v/c)$ terms from (95.84). Then if we use (96.7), we will obtain satisfactory results in the optically thin regime, but will make serious errors in the optically thick regime, where $J \rightarrow B$ and the right-hand side vanishes almost identically, because we have not accounted explicitly for either the rate of change of the internal energy in the radiation or the rate of work done by radiation pressure. Castor concludes (**C3**) that in the diffusion regime the temperature determined from (96.7) with the $O(v/c)$ terms omitted from (95.84) can be in error by an amount of $O(P/p)$. If, instead, we use (96.9) the difficulty is reversed. We then obtain an accurate solution at great depth, but will make serious errors in the optically thin regime where the gas decouples from the radiation; Castor finds that the error in the temperature is again $O(P/p)$. In short, it is *essential* to retain $O(v/c)$ terms in (95.84) in order to bridge the transition between the optically thick and thin limits.

The situation for the momentum equation is different. Here (DF_0/Dt) and the velocity-dependent terms multiplying F_0 in (96.3) are never larger than $O(v/c)$, and are much smaller in the diffusion limit. We can therefore drop these terms, which means that we will obtain consistency with (96.2) even if we drop the time-derivative and velocity-dependent terms from the radiation momentum equation (95.85). Moreover, in the derivation of the mechanical energy equation (96.12), which when combined with (96.9), leads to the total energy equation (96.15), all $O(v/c)$ terms in (95.85) become $O(v^2/c^2)$, and hence can be dropped from the outset. In short, we do not adversely affect consistency among various forms of the energy or momentum equations by dropping all $O(v/c)$ terms from (95.85).

CONSISTENCY OF THE INERTIAL-FRAME AND COMOVING-FRAME ENERGY AND MOMENTUM EQUATIONS FOR A RADIATING FLUID

Let us now examine the mutual consistency of the inertial-frame and comoving-frame energy and momentum equations. Consider first the inertial-frame gas-energy equation (94.19b). On a fluid-flow time scale the $(\partial \mathbf{F}/\partial t)$ term is $O(v^2/c^2)$ relative to $\nabla \cdot \mathbf{F}$ and hence can be dropped. Similarly the (v/c) terms in the transformations of (E, P) into (E_0, P_0) will produce terms of $O(v^2/c^2)$; we thus need to retain $O(v/c)$ terms only to transform $\mathbf{F}$ to $\mathbf{F}_0$. In particular, for one-dimensional spherically symmetric flow we have

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2(F_0 + vE_0 + vP_0)]$$
$$= \frac{\partial F_0}{\partial r} + \frac{2F_0}{r} + v\left(\frac{\partial E_0}{\partial r} + \frac{\partial P_0}{\partial r}\right) + \left(\frac{\partial v}{\partial r} + \frac{2v}{r}\right)(E_0 + P_0). \tag{96.19}$$

Furthermore, from (66.10)

$$\nabla \cdot \mathsf{P}_0 = (\partial P_0/\partial r) + (3P_0 - E_0)/r. \tag{96.20}$$

Using these results in (94.19b) we find

$$\rho\left[\frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = -\left[\frac{DE_0}{Dt} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_0) + \frac{v}{r}(3E_0 - P_0) + (E_o + P_0)\frac{\partial v}{\partial r}\right], \tag{96.21}$$

which, by virtue of (95.84) is identical to the comoving-frame gas-energy equation (96.7). If the velocity-dependent term on the right-hand side of (94.19b) had been omitted, we would be left with an extra term in (96.21) of the form $v(\partial P/\partial r)$, that is, the rate of work done by the fluid against the radiation pressure gradient. For fluids with intense radiation fields, this term is large and would lead to serious errors. By a similar analysis, one readily shows that (94.22) is consistent with (96.9).

Alternatively, consider the inertial-frame equation (94.19a), which for grey material reduces to

$$\rho\left[\frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \kappa_0(cE - 4\pi B_0 - 2\mathbf{v}\cdot\mathbf{F}/c) + O(v^2/c^2). \tag{96.22}$$

Then using (91.16) we have

$$\rho\left[\frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \kappa_0(cE_0 - 4\pi B_0), \tag{96.23}$$

which is identical to the comoving-frame equation (96.7) for grey material. Had the $O(v/c)$ terms been omitted from (94.19a), from (93.10) and (93.11), or from (91.16), this exact reduction would not be achieved; the error would equal $\kappa_0 \mathbf{v} \cdot \mathrm{F}/c$, the rate of work done by radiation forces on the material.

In summary, consistency between the inertial-frame and comoving-frame equations requires that all $O(v/c)$ terms be retained in both gas-energy equations, in the radiation energy equation, and in the transformation laws between frames [see also (**P4**)]. In contrast, all $O(v/c)$ terms can be omitted from the radiation momentum equation without loss of consistency.

Finally, consider the inertial-frame momentum equation (94.13b), which for spherically symmetric flow reduces to

$$\rho\frac{Dv}{Dt}=\frac{-GM_r\rho}{r^2}-\frac{\partial p}{\partial r}-\left[\frac{1}{c^2}\frac{\partial F}{\partial t}+\frac{\partial P}{\partial r}+\frac{(3P-E)}{r}-\frac{v}{c^2}\left(\frac{\partial E}{\partial t}+\frac{\partial F}{\partial r}+\frac{2F}{r}\right)\right]. \tag{96.24}$$

On a fluid-flow time scale the term containing $(\partial E/\partial t)$ is $O(v^2/c^2)$ relative to $(\partial P/\partial r)$, and therefore can be dropped. Similarly all terms containing F are at most $O(v/c)$ relative to the terms in E and P. Hence to obtain a final result accurate to $O(v/c)$ it is sufficient to set $F=F_0$, but all terms must be retained in transforming from (E, P) to (E_0, P_0). Making these conversions we find

$$\rho\frac{Dv}{Dt}=\frac{-GM_r\rho}{r^2}-\frac{\partial p}{\partial r}-\left[\frac{1}{c^2}\frac{\partial F_0}{\partial t}+\frac{v}{c^2}\frac{\partial F_0}{\partial r}+\frac{\partial P_0}{\partial r}+\frac{(3P_0-E_0)}{r}+\frac{2}{c^2}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)F_0\right], \tag{96.25}$$

which is identical to the comoving-frame equation (96.3). Thus consistency of the momentum equation between frames is assured if, and only if, one accounts for $O(v/c)$ terms in both frames.

Similarly, in light of (93.10) and (93.11) the inertial-frame momentum equation (94.13a) for a spherically symmetric flow of grey material is

$$\rho(Dv/Dt)=-(GM_r\rho/r^2)-(\partial p/\partial r)+(\kappa_0/c)[F-(v/c)(E+P)]+O(v^2/c^2), \tag{96.26}$$

which, from (91.19), is identical to the comoving-frame equation (96.2) for grey material. Again we see that the $O(v/c)$ terms are essential for consistency.

7.3 Solution of the Equations of Radiation Hydrodynamics

MATHEMATICAL STRUCTURE OF THE PROBLEM

In §§93 to 96 we formulated the equations of radiation hydrodynamics in both the Eulerian and Lagrangean frames; we now ask how to solve them. In this connection it is instructive to count the number of variables to be determined and the number of equations available to determine them, as in §24. As before we must find seven fluid variables: ρ, p, T, e, and three components of $\mathbf{v}$; in addition we must now find ten radiation variables: E, the three components of $\mathbf{F}$, and the six nonredundant components of P.